Chapter 6

EXACT MULTIVARIATE TESTS OF ASSET PRICING MODELS WITH STABLE ASYMMETRIC DISTRIBUTIONS

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Abstract In this chapter, we propose exact inference procedures for asset pricing models that can be formulated in the framework of a multivariate linear regression (CAPM), allowing for stable error distributions. The normality assumption on the distribution of stock returns is usually rejected in empirical studies, due to excess kurtosis and asymmetry. To model such data, we propose a comprehensive statistical approach which allows for alternative — possibly asymmetric — heavy tailed distributions without the use of large-sample approximations. The methods suggested are based on Monte Carlo test techniques. Goodness-of-fit tests are formally incorporated to ensure that the error distributions considered are empirically sustainable, from which exact confidence sets for the unknown tail area and asymmetry parameters of the stable error distribution are derived. Tests for the efficiency of the market portfolio (zero intercepts) which explicitly allow for the presence of (unknown) nuisance parameter in the stable error distribution are derived. The methods proposed are applied to monthly returns on 12 portfolios of the New York Stock Exchange over the period 1926 – 1995 (5 year subperiods). We find that stable possibly skewed distributions provide statistically significant improvement in goodness-of-fit and lead to fewer rejections of the efficiency hypothesis.

1. Introduction

An important problem in empirical finance consists in testing the efficiency of a market portfolio by assessing the statistical significance of the intercepts of a multivariate linear regression (MLR) on asset returns (the capital asset pricing model (CAPM)); see MacKinlay (1987), Jobson and Korkie (1989), Gibbons et al. (1989), Shanken (1996), Campbell et al. (1997, Chapters 5 and 6), Stewart (1997), and Fama and French (2003). Traditional statistical theory supplies a reliable distributional theory mainly in the case where the disturbances in the model follow a Gaussian distribution; see, for example, Anderson (1984, Chapters 8 and 13) and Rao (1973, Chapter 8). However, in financial data, the Gaussian assumption is typically inappropriate, because asset returns often exhibit excess kurtosis and asymmetries; see, for example, Fama (1965), Baillie and Bollerslev (1989), Beaulieu (1998), and Dufour et al. (2003). Further, asymptotic approximations aimed at relaxing the Gaussian assumption tend to be unreliable in multivariate models such as those considered in CAPM applications, especially when the number of equations (or assets) is not small; see Campbell et al. (1997, Chapter 5), Gibbons et al. (1989), Shanken (1996, Section 3.4.2), and Dufour and Khalaf (2002b). Consequently, it is important from an inference viewpoint that we approach this problem from a finite sample perspective.[1]

In recent work (Dufour et al., 2003; Beaulieu et al., 2004), we considered this problem by developing exact efficiency tests of the market portfolio in the case where the CAPM disturbances follow t distributions or normal mixtures. In particular, we observed that: (i) monthly returns reject multivariate normality conclusively, and (ii) CAPM tests based on the assumption of elliptical errors yield less rejections than those based on the (erroneous) normality assumption. The latter result obtains if the (unknown) parameters underlying the elliptical error distribution are formally accounted for.[2] Indeed, the whole issue centers on the uncertainty associated with unknown (nuisance) parameters, one of the main difficulties which complicate the development of exact tests. This analysis was however restricted to symmetric error distributions.

In the present chapter, we consider distributional models that can accommodate more pronounced skewness and kurtosis. Specifically, we study the case where the disturbances in a CAPM regression can follow stable possibly asymmetric distributions. Our results reveal notable dif-

[1]For more general discussions of the importance of developing finite-sample statistical procedures, see Dufour (1997, 2003).

[2]Concerning normality tests, our procedures achieve size control exactly, so test rejections cannot be spurious by construction. Concerning tests on intercepts, we formally demonstrate location-scale invariance of the commonly used procedures for the context at hand. Since the normal distribution is completely defined by its mean and variance, nuisance parameter-free test procedures can easily be derived. Non-normal distributions raise further nuisance parameter problems; examples include the number of degrees of freedom, for a multivariate Student t distribution, and the probability-of-mixing and scale-ratio parameters for normal mixtures.

ferences with respect to the mainstream elliptical framework. Besides being consistent with optimization arguments underlying the CAPM (see Samuelson, 1967), the family of stable distributions is entailed by various central limit arguments in probability theory (as an alternative to the Gaussian distribution) and has often been suggested as a useful model for return and price distributions in finance; see, for example, Mandelbrot (1963), Ibragimov and Linnik (1975), Zolotarev (1986), Cambanis et al. (1991), Samorodnitsky and Taqqu (1994), Embrechts et al. (1997), Rachev et al. (1999a,b), Uchaikin and Zolotarev (1999), Adler et al. (2000), Mittnik et al. (2000), Rachev and Mittnik (2000), and Meerschaert and Scheffler (2001). One should note, however, that tests and confidence sets which have been proposed for inference on such models are almost always based on asymptotic approximations that can easily be unreliable. Further, standard regularity conditions and asymptotic distributional theory may easily not apply to such distributions (for example, because of heavy tails).

To obtain finite-sample inference for such models, we combine several techniques. *First*, we obtain finite-sample joint confidence sets for the unknown parameters of the stable distribution (i.e., the tail thickness α_s and the asymmetry β_s) through the "inversion" of goodness-of-fit tests based on multivariate kurtosis and skewness coefficients computed from model residuals. *Second*, in view of the complicated distribution of these statistics, we exploit invariance properties of the goodness-of-fit statistics to implement the corresponding tests as finite-sample Monte Carlo (MC) tests (as proposed in Dufour et al., 2003). *Thirdly*, using general results from Dufour and Khalaf (2002b) on hypothesis testing in multivariate linear regressions with non-Gaussian disturbances, we note that finite-sample standard LR-type efficiency tests can easily be obtained as soon as the parameters (α_s, β_s) of the stable error distribution are specified, again through the application of the MC test technique. *Fourth*, we exploit a two-stage confidence technique proposed in Dufour (1990), Dufour and Kiviet (1996, 1998), and Dufour et al. (1998b) to derive efficiency tests that formally take into account the uncertainty of the stable distribution parameters (α_s, β_s) by maximizing the MC p-values associated with different nuisance parameter values (α_s, β_s) over a confidence set for the latter built as described in the first step above (with an appropriately selected level).

The technique of MC tests — which plays a crucial role in our approach — is an exact simulation-based inference procedure originally proposed by Dwass (1957) and Barnard (1963). It is related to the parametric bootstrap in the sense that the distribution of the test statistic is simulated under the null hypothesis. When the latter does not involve unknown nuisance parameters, the MC test method controls the size of the procedure perfectly, while bootstrap methods are justified only by asymptotic arguments. The finite-sample theory that underlies MC tests allows one to implement test statistics with very complicated distributions (as long as they can be simulated) and does not require establishing a limit distribution as the sample size goes to infinity (or even the existence of such a distribution). It is easy to see that this feature can be quite convenient when dealing with stable distributions under which standard central limit theorems may not apply. The contrast is even more important when test statistics involve nuisance parameters. Here we use extensions of this MC test technique that allow for the presence of nuisance parameters. The level of the test can be controlled in finite samples as soon as the null distribution of the test statistic can be simulated once the values of the nuisance parameters are set.[3] This is clearly not the case in bootstrapping, where bootstrap samples are drawn after setting the unknown nuisance parameters at some "consistent" estimate. For further discussion of Monte Carlo test methods, see, for example, Dufour (2002), Dufour and Khalaf (2001, 2002a,b, 2003), Dufour and Kiviet (1996, 1998), Kiviet and Dufour (1997), Dufour et al. (1998a, 2004, 2003), and Beaulieu et al. (2004). Since bootstrap-type procedures are gaining popularity in finance (see, e.g., Li and Maddala, 1996), we emphasize the importance of using such procedures correctly.

We show that the proposed approach is both practical and useful from an empirical viewpoint by applying it to monthly returns on 12 portfolios of the New York Stock Exchange over the period 1926 − 1995 (5 year subperiods). Among other things we find that heavy-tailed skewed distributions provide statistically significant improvement in goodness-of-fit and lead to fewer rejections of the efficiency hypothesis. Our results show clearly that the introduction of an asymmetric distribution instead of an elliptical distribution yields noteworthy changes in the decision regarding the efficiency hypothesis of the market portfolio. In our opinion this is an important finding since CAPM rejections are often attributed to the presence of excess kurtosis in stock returns. Further, inference on the tail thickness parameter α_s appears to be more precise than inference on the asymmetry parameter β_s.

The chapter is organized as follows. Section 2 describe the model and test problem studied. In Section 3, we describe the existing test procedures and we show how extensions allowing for nonnormal distributions

[3] In nuisance parameter dependent problems, a test is *exact* at level α if the largest rejection probability over the nuisance parameter space consistent with the null hypothesis is not greater than α (see Lehmann, 1986, Sections 3.1 and 3.5).

are obtained. In Section 4 we report the empirical results. Section 5 concludes and discusses extensions to other asset pricing tests.

2. Framework

The framework we consider here is the same one as in Beaulieu et al. (2004):

$$r_{it} = a_i + b_i\tilde{r}_{\mathrm{M}t} + u_{it}, \quad t = 1, \ldots, T, i = 1, \ldots, n, \tag{6.1}$$

where $r_{it} = R_{it} - R_t^F$, $\tilde{r}_{\mathrm{M}t} = \tilde{R}_{\mathrm{M}t} - R_t^F$, R^F is the riskless rate of return, R_{it}, $i = 1, \ldots, n$, are returns on n securities for period t, $\tilde{R}_{\mathrm{M}t}$ is the return on the market portfolio, and u_{it} is a random disturbance.[4] In this context, the CAPM entails the following efficiency restrictions:

$$H_{\mathrm{CAPM}} : a_i = 0, \quad i = 1, \ldots, n, \tag{6.2}$$

i.e., the intercepts a_i are jointly equal to zero (Gibbons et al., 1989).

The above model can be cast in matrix form as a MLR model:

$$Y = XB + U \tag{6.3}$$

where $Y = [Y_1, \ldots, Y_n]$ is a $T \times n$ matrix of dependent variables, X is a $T \times k$ full-column rank matrix of regressors, and

$$U = [U_1, \ldots, U_n] = [V_1, \ldots, V_T]' \tag{6.4}$$

is a $T \times n$ matrix of random disturbances. Specifically, to get (6.1), we set:

$$Y = [r_1, \ldots, r_n], \quad X = [\iota_T, \tilde{r}_{\mathrm{M}}], \quad \iota_T = (1, \ldots, 1)', \tag{6.5}$$

$$r_i = (r_{1i}, \ldots, r_{Ti})', \quad \tilde{r}_{\mathrm{M}} = (\tilde{r}_{1\mathrm{M}}, \ldots, \tilde{r}_{T\mathrm{M}})'. \tag{6.6}$$

Further, in the matrix setup, the mean-variance efficiency restriction H_{CAPM} belongs to the class of so-called *uniform linear* (UL) restrictions, i.e., it has the form

$$H_0 : HB = D \tag{6.7}$$

where H is an $h \times k$ matrix of rank h. H_{CAPM} corresponds to the case where $h = 1$, $H = (1, 0)$ and $D = 0$.

In general, asset pricing models impose further restrictions on the error distributions. In particular, the standard CAPM obtains assuming that

$$V_1, \ldots, V_T \overset{\text{i.i.d.}}{\sim} N[0, \Sigma] \tag{6.8}$$

[4]For convenience, we focus here on the single beta case. For some discussion of the multi-beta CAPM, see Beaulieu et al. (2004).

or elliptically symmetric (Ingersoll, 1987); for recent references, see Hodgson et al. (2002), Vorkink (2003), Hodgson and Vorkink (2003), and the references cited therein. We consider the more general case

$$V_t = JW_t, \quad t = 1, \ldots, T, \tag{6.9}$$

where J is an unknown nonsingular matrix, $W_t = (W_{1t}, \ldots, W_{nt})'$ is a $n \times 1$ random vector, and the distribution of $w = \text{vec}(W_1, \ldots, W_T)$ conditional on X is either: (i) completely specified (hence, free of nuisance parameters), or (ii) partially specified up to an unknown nuisance-parameter. We call w the vector of *normalized disturbances* and its distribution the *normalized disturbance distribution.* When W_t has an identity covariance matrix, i.e.,

$$\mathsf{E}[W_t W_t'] = I_n, \tag{6.10}$$

the matrix $\Sigma = JJ'$ is the covariance matrix of V_t, so that $\det(\Sigma) \neq 0$. Note that the assumption (6.10) will not be needed in the sequel. No further regularity conditions are required for most of the statistical procedures proposed below, not even the existence of second moments.

In Beaulieu et al. (2004), we focused on multivariate t distributions and normal mixtures, which we denote $\mathcal{F}_1(W)$ and $\mathcal{F}_2(W)$ respectively, and define as follows:

$$W \sim \mathcal{F}_1(W;\kappa) \iff W_t = Z_{1t}/(Z_{2t}/\kappa)^{1/2}, \tag{6.11}$$

where Z_{1t} is multivariate normal $(0, I_n)$ and Z_{2t} is a $\chi^2(\kappa)$ variate independent from Z_{1t};

$$W \sim \mathcal{F}_2(W;\pi,\omega) \iff W_t = \pi Z_{1t} + (1-\pi)Z_{3t}, \tag{6.12}$$

where Z_{3t} is multivariate normal $(0, \omega I_n)$ and is independent from Z_{1t}, and $0 < \pi < 1$.

In the present chapter, we extend our empirical investigation to asymmetric stable distributions

$$W \sim \mathcal{F}_s(W;\alpha_s,\beta_s) \iff W_{ti} \overset{\text{i.i.d.}}{\sim} S(\alpha_s,\beta_s), \quad i = 1, \ldots, n, \tag{6.13}$$

where $S(\alpha_s, \beta_s)$ represents the stable distribution with the tail thickness α_s, skewness parameter β_s, location parameter zero and scale parameter one. In view of the presence of a regression model (6.1) and the J matrix in (6.9), the location and scale parameters of W_t can be set to zero and one without loss of generality (and for identification purposes). As it is well known, a simple closed-form expression is not available for stable distributions (except in special cases) but there is one for the characteristic function $\phi(t)$: if $S \sim S(\alpha_s, \beta_s)$,

$$\ln \phi(t) = \ln \mathsf{E}[\exp(itS)] = \begin{cases} -|t|^{\alpha_s}[1 - i\beta_s \operatorname{sgn}(t) \tan(\pi\alpha_s/2)], & \text{for } \alpha_s \neq 1, \\ -|t|[1 + i\beta_s(2/\pi) \operatorname{sgn}(t) \ln |t|], & \text{for } \alpha_s = 1, \end{cases}$$

where $0 < \alpha_s \leq 2$ and $-1 \leq \beta_s \leq 1$, and sgn(t) is the sign function, i.e.,

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0; \end{cases} \tag{6.14}$$

see Rachev and Mittnik (2000, Chapter 2), Samorodnitsky and Taqqu (1994, Chapter 1). Note also that random variables with stable distributions can easily be simulated; see Chambers et al. (1976) and Weron (1996).

For further reference, we use the following notation:

$$W \sim \mathcal{F}_i(W; \nu), i = 1, 2, \tag{6.15}$$

where ν is the vector of nuisance parameters in the distribution of W, for example

$$\begin{aligned} \nu = \kappa, \quad & \text{if } W_t \text{ satisfies (6.11)}, \\ = (\pi, \omega), \quad & \text{if } W_t \text{ satisfies (6.12)}, \\ = (\alpha_s, \beta_s), \quad & \text{if } W_t \text{ satisfies (6.13)}. \end{aligned}$$

In the sequel, we shall focus on the third case where $\nu = (\alpha_s, \beta_s)$ may be unknown.[5]

3. Statistical method

As in Gibbons et al. (1989), the statistic we use to test H_{CAPM} in (6.2) is the Gaussian quasi maximum likelihood (QMLE) based criterion:

$$\text{LR} = T \ln(\Lambda), \quad \Lambda = |\widehat{\Sigma}_{\text{CAPM}}|/|\widehat{\Sigma}|, \tag{6.16}$$

where $\widehat{\Sigma} = \widehat{U}'\widehat{U}/T$, $\widehat{U} = Y - X\widehat{B}$, $\widehat{B} = (X'X)^{-1}X'Y$ and $\widehat{\Sigma}_{\text{CAPM}}$ is the Gaussian QMLE under H_{CAPM}. In Beaulieu et al. (2004), we derive the

[5] For a theoretical discussion of the CAPM with stable Paretian laws, see Samuelson (1967). For discussions of the class of return distributions compatible with the CAPM, see Ross (1978); Chamberlain (1983), Ingersoll (1987, Chapter 4), Nielsen (1990), Allingham (1991), Berk (1997) and Dachraoui and Dionne (2003).

exact null distribution of the latter statistic under (6.1) and (6.9). This result is reproduced here for convenience.

THEOREM 6.1 *Under* (6.1)*,* (6.2) *and* (6.9)*, the* LR *statistic defined by* (6.16) *is distributed like*

$$T\ln(|W'MW|/|W'M_0W|), \tag{6.17}$$

where

$$M = I - X(X'X)^{-1}X', \tag{6.18}$$

$$M_0 = M + X(X'X)^{-1}H'[H(X'X)^{-1}H']^{-1}H(X'X)^{-1}X', \tag{6.19}$$

$$X = [\iota_T, \tilde{r}_{\mathrm{M}}], \quad \tilde{r}_{\mathrm{M}} = (\tilde{r}_{1\mathrm{M}}, \ldots, \tilde{r}_{T\mathrm{M}})', \tag{6.20}$$

H *is the row vector* $(1,0)$*, and* $W = [W_1, \ldots, W_T]'$ *is defined by* (6.9)*.*

We exploit two results regarding this distribution, the first one being a special case of the latter. First, Theorem 6.1 leads to Gibbons et al.'s (1989) results. Specifically, when errors are Gaussian,

$$\frac{T-s-n}{n}(\Lambda - 1) \sim F(n, T-s-n),$$

which yields Hotelling's T^2 test proposed by MacKinlay (1987) and Gibbons et al. (1989). Second, under the general assumption (6.9), the null distribution of (6.16) does not depend on B and Σ and may thus easily be simulated if draws from the distribution of $W_1, \ldots, W_T$ are available. This entails that a Monte Carlo exact test procedure (Dufour, 2002) may be easily applied based on LR. The general simulation-based algorithm which allows to obtain a MC size-correct exact p-value for all hypotheses conforming with (6.9) and (6.15) is presented in Beaulieu et al. (2004) and may be summarized as follows.

Given ν in (6.15), generate N *i.i.d.* draws from the distribution of $W_1, \ldots, W_T$; on applying (6.17), these yield N simulated values of the test statistic. The exact Monte Carlo p-value is then calculated from the rank of the observed LR [denoted by LR_0] relative to the simulated ones:

$$\hat{p}_N(\mathrm{LR}_0 \mid \nu) = \frac{N\widehat{G}_N(S_0)+1}{N+1} \tag{6.21}$$

where $N\widehat{G}_N(\mathrm{LR}_0)$ is the number of simulated criteria not smaller than LR_0.

In Beaulieu et al. (2004) we also consider testing H_{CAPM} (6.2) in the context of

$$r_{it} = a_i + \sum_{j=1}^{s} b_{ji}\tilde{r}_{jt} + u_{it}, \quad t = 1, \ldots, T, \; i = 1, \ldots, n, \tag{6.22}$$

where $\tilde{r}_{jt} = \tilde{R}_{jt} - R^F_t$ and $\tilde{R}_{jt}$, $j = 1, \ldots, s$, are returns on s benchmark portfolios. In this case, the null distribution of the statistic defined by (6.16) obtains as in Theorem 6.1 where

$$X = [\iota_T, \tilde{r}_1, \ldots, \tilde{r}_s], \quad \tilde{r}_j = (\tilde{r}_{1j}, \ldots, \tilde{r}_{Tj})' \tag{6.23}$$

and H is the $(s+1)$-dimensional row vector $(1, 0, \ldots, 0)$.

Let us now extend the above results to the unknown distributional parameter case for the error families of interest, namely (6.15). The α-level procedure adopted in Beaulieu et al. (2004) (based on Dufour 1990 and Dufour and Kiviet 1996) involves two stages: (1) build an exact confidence set (denoted $\mathcal{C}(Y)$) for ν, with level $1 - \alpha_1$; (2) maximize the p-value function $\hat{p}_N(LR_0|\nu)$ in (6.21) over-all values of ν in the latter confidence set; then compare the latter maximal p-value with α_2 where $\alpha = \alpha_1 + \alpha_2$.[6] Formally, the test we denote *maximized MC* (MMC) *test*, is significant if

$$Q_U(\nu) \leq \alpha_2 \tag{6.24}$$

where

$$Q_U(\nu) = \sup_{\nu \in \mathcal{C}(Y)} \hat{p}_N(LR_0 \mid \nu). \tag{6.25}$$

To obtain $\mathcal{C}(Y)$, we proceed by "inverting" a goodness-of-fit (GF) test for the null hypothesis (6.15) where $\nu = \nu_0$ for known ν_0, as proposed in Dufour et al. (2003). The GF test statistic is based on the following excess skewness and kurtosis criteria:

$$\mathrm{ESK}(\nu_0) = |\,\mathrm{SK} - \overline{\mathrm{SK}}(\nu_0)|, \tag{6.26}$$

$$\mathrm{EKU}(\nu_0) = |\,\mathrm{KU} - \overline{\mathrm{KU}}(\nu_0)|, \tag{6.27}$$

where SK and KU are the well known multivariate measures (see Mardia, 1970):

$$\mathrm{SK} = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{i=1}^{T} \hat{d}_{ii}^3, \tag{6.28}$$

$$\mathrm{KU} = \frac{1}{T} \sum_{t=1}^{T} \hat{d}_{tt}^2, \tag{6.29}$$

$\hat{d}_{it}$ are the elements of the matrix $\widehat{D} = \widehat{U}(\widehat{U}'\widehat{U})^{-1}\widehat{U}'$ and $\overline{\mathrm{SK}}(\nu_0)$ and $\overline{\mathrm{KU}}(\nu_0)$ are simulation-based estimates of the expected SK and KU given

[6] In the empirical section, we use $\alpha_1 = \alpha_2 = \alpha/2$.

by (6.15). Given ν_0, these may be obtained by drawing N_0 samples of T observations from (6.15), and then computing the corresponding average measures of skewness and kurtosis.[7] Specifically, we use the combined criterion

$$\text{CSK} = 1 - \min\big\{\hat{p}\big(\text{ESK}(\nu_0) \mid \nu_0), \hat{p}(\text{EKU}(\nu_0) \mid \nu_0)\big)\big\}, \tag{6.30}$$

where $\hat{p}_N(\text{ESK}(\nu_0) \mid \nu_0)$ and $\hat{p}_N(\text{EKU}(\nu_0) \mid \nu_0)$ are MC p-values based on $\text{ESK}(\nu_0)$ and $\text{EKU}(\nu_0)$.[8] The intuition underlying this combined criterion is to reject the null hypothesis if at least one of the individual tests is significant; for convenience, we subtract the minimum p-value from one to obtain a right-sided test. The MC test technique is once again applied to obtain a test based on the combined statistic; details of the algorithm can be found in Dufour et al. (2003) and Beaulieu et al. (2004). For further reference on such combined tests, see Dufour and Khalaf (2002a) and Dufour et al. (2004).

4. Empirical analysis

Our empirical analysis focuses on testing (6.2) in the context of (6.1) with different distributional assumptions on stock market returns. We use nominal monthly returns over the period going from January 1926 to December 1995, obtained from the University of Chicago's Center for Research in Security Prices (CRSP). As in Breeden et al. (1989), our data include 12 portfolios of New York Stock Exchange (NYSE) firms grouped by standard two-digit industrial classification (SIC). Table 6.1 provides a list of the different sectors used as well as the SIC codes included in the analysis.[9] For each month the industry portfolios comprise those firms for which the return, price per common share and number of shares outstanding are recorded by CRSP. Furthermore, portfolios are value-weighted in each month. In order to assess the testable implications of the asset pricing models, we proxy the market return with the value-weighted NYSE returns, also available from CRSP. The risk-free rate is proxied by the one-month Treasury Bill rate, also from CRSP.

Our results are summarized in Tables 6.2 – 6.4. All MC tests were applied with $N = 999$ replications. As usual in this literature, we es-

[7] For the Gaussian case, one may use $\overline{\text{SK}} = 0$ and $\overline{\text{KU}} = n(n+2)$; see Mardia (1970).

[8] In Beaulieu et al. (2004), we demonstrate that these criteria are pivotal, i.e., under (6.15), their null distribution does not depend on B and Σ and thus may easily be simulated if draws from the distribution of $W_1, \ldots, W_T$ are available. Hence the MC p-values $\hat{p}_N\big(\text{ESK}(\nu_0) \mid \nu_0\big)$ and $\hat{p}_N\big(\text{EKU}(\nu_0) \mid \nu_0\big)$ can be obtained following the same simulation technique underlying $\hat{p}_N(\text{LR}_0 \mid \nu)$; see (6.21).

[9] As in Breeden et al. (1989), firms with SIC code 39 (Miscellaneous manufacturing industries) are excluded from the dataset for portfolio formation.

Table 6.1. Portfolio definitions

Portfolio number	Industry Name	Two-digit SIC codes
1	Petroleum	13, 29
2	Finance and real estate	60 – 69
3	Consumer durables	25, 30, 36, 37, 50, 55, 57
4	Basic industries	10, 12, 14, 24, 26, 28, 33
5	Food and tobacco	1, 20, 21, 54
6	Construction	15 – 17, 32, 52
7	Capital goods	34, 35, 38
8	Transportation	40 – 42, 44, 45, 47
9	Utilities	46, 48, 49
10	Textile and trade	22, 23, 31, 51, 53, 56, 59
11	Services	72, 73, 75, 80, 82, 89
12	Leisure	27, 58, 70, 78, 79

Note. This table presents portfolios according to their number and sector as well as the SIC codes included in each portfolio using the same classification as Breeden et al. (1989).

timate and test the model over intervals of 5 years.[10] In columns (1), (2), (6), (8) and (9) of Table 6.2, we present the LR and its asymptotic $\chi^2(n)$ p-value (p_∞), and stable errors based on maximal MC p-values (Q_U). For comparison purposes, we also report [in columns (3) – (4)] the Gaussian based MC p-value $p_{\mathcal{N}}$ and the Student t MMC p-value (Q_U) from Beaulieu et al. (2004). The confidence sets $\mathcal{C}(Y)$ for the nuisance parameters appear in columns (5), (7) and (10). To simplify the presentation, the confidence region is summarized as follows: we present the confidence sets for α_s given $\beta_s = 0$, and the union of the confidence sets for α_s given $\beta_s \neq 0$. These results allow one to compare rejection decisions across different distributional assumptions for the returns of the 12 portfolios.

Our empirical evidence shows the following. In general, asymptotic p-values are quite often spuriously significant (e.g., 1941 – 55). Furthermore, non-Gaussian based maximal p-values exceed the Gaussian-based p-value. Note however that the results of exact goodness-of-fit tests (available from Dufour et al., 2003) indicate that normality is definitively rejected except in 1961 – 65 and 1991 – 95.

As emphasized in Beaulieu et al. (2004), it is "easier" to reject the testable implications under normality, and any symmetric error considered. Indeed, at the 5% significance level, we find ten rejections of the

[10] Note that we also ran the analysis using ten year subperiods and that our results were not significantly affected.

Table 6.2. CAPM tests

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				Student t		Stable symmetric $\beta_s = 0$		Stable asymmetric		
								$\beta_s > 0$	$\beta_s < 0$	$\beta_s \neq 0$
Sample	QLR	p_∞	$p_\mathcal{N}$	Q_U	$C(Y)$	Q_U	$C(Y)$	Q_U	Q_U	$C(Y)$
1927 – 30	16.10	.187	.364	.357	3 – 12	.367	1.38 – 1.96	.927	.941	1.44 – 1.94
1931 – 35	16.26	.180	.313	.322	3 – 8	.298	1.34 – 1.92	.926	.925	1.42 – 1.92
1936 – 40	16.02	.190	.319	.333	4 – 26	.316	1.56 – 1.98	.737	.764	1.56 – 1.98
1941 – 45	25.87	.011	.045	.049	≥ 5	.031	1.58 – 1.98	.324	.285	1.56 – 1.98
1946 – 50	37.20	.000	.003	.004	4 – 26	.002	1.56 – 1.98	.108	.082	1.56 – 1.98
1951 – 55	36.51	.000	.004	.005	5 – 31	.001	1.56 – 1.98	.084	.048	1.56 – 1.98
1956 – 60	43.84	.000	.002	.002	≥ 5	.001	1.56 – 1.98	.032	.014	1.58 – 1.98
1961 – 65	39.10	.000	.002	.002	≥ 7	.001	1.66 – 2.00	.044	.020	1.20 – 1.99
1966 – 70	36.79	.000	.003	.003	≥ 5	.001	1.56 – 1.98	.116	.044	1.58 – 1.99
1971 – 75	21.09	.049	.120	.129	4 – 24	.111	1.56 – 1.98	.566	.596	1.56 – 1.98
1976 – 80	28.37	.005	.023	.026	4 – 17	.017	1.50 – 1.98	.425	.329	1.50 – 1.98
1981 – 85	27.19	.007	.033	.035	5 – 34	.023	1.56 – 1.98	.324	.309	1.56 – 1.98
1986 – 90	35.75	.001	.003	.005	≥ 5	.004	1.62 – 2.00	.086	.058	1.63 – 1.99
1991 – 95	16.75	.159	.299	.305	≥ 15	.287	1.68 – 2.00	.473	.477	1.70 – 1.99

Note. Column (1) presents the quasi-LR statistic defined in (6.16) to test H_{CAPM} (see (6.2)); columns (2), (3), (4), (6), (8) and (9) are the associated p-values using, respectively, the asymptotic $\chi^2(n)$ distribution, the pivotal statistics based MC test method imposing multivariate normal regression errors, an MMC confidence set based method imposing, in turn, multivariate $t(\kappa)$ errors, symmetric stable and asymmetric stable errors, which yields the largest MC p-value for all nuisance parameters within the specified confidence sets. The latter are reported in columns (5), (7) and (10); for convenience, for the asymmetric stable case, we present the union of the confidence sets for α_s given $\beta_s \neq 0$. October 1987 and January returns are excluded from the dataset.

null hypothesis for the asymptotic $\chi^2(11)$ test, nine for the MC p-values under normality, eight under a symmetric stable error distribution, and just two rejections (1956 – 60, 1961 – 65) with left-skewed (negative β_s) asymmetric stable errors; no rejections are noted with right-skewed (positive β_s) asymmetric stable errors. Note that our MC tests under non-normal errors are joint tests for nuisance parameters consistent with the data and the efficiency hypothesis. Since we used $\alpha_1 = 0.025$ for the construction of the confidence set, to establish a fair comparison with the MC p-values under the normality assumption or the asymptotic p-values, we must refer the p-values for the efficiency tests under the Student and

Table 6.3. Supremum p-values for various positive skewness measures

β_s	0	.3	.4	.5	.6	.7	.9	.99
1927 – 30	.367	.540	.665	.777	.759	.798	.888	.927
1931 – 35	.298	.549	.640	.744	.876	.919	.907	.926
1936 – 40	.316	.395	.456	.521	.538	.639	.688	.737
1941 – 45	.031	.052	.070	.096	.129	.170	.276	.324
1946 – 50	.002	.004	.006	.010	.017	.034	.080	.108
1951 – 55	.001	.003	.004	.007	.018	.030	.058	.084
1956 – 60	.001	.002	.002	.002	.003	.006	.020	.032
1961 – 65	.002	.002	.002	.002	.003	.008	.017	.044
1966 – 70	.001	.002	.009	.010	.021	.034	.080	.116
1971 – 75	.011	.154	.199	.246	.299	.362	.490	.566
1976 – 80	.017	.033	.063	.106	.166	.197	.418	.425
1981 – 85	.023	.043	.052	.079	.116	.164	.277	.324
1986 – 90	.004	.006	.010	.013	.019	.022	.063	.086
1991 – 95	.287	.296	.307	.324	.358	.388	.443	.473

Note. Numbers shown are p-values associated with our efficiency test using an MMC confidence set based method imposing asymmetric stable errors, which yields, given the specific $\beta_s > 0$, the largest MC p-value for all α_s within the specified confidence sets. The latter are reported in Table 6.2. October 1987 and January returns are excluded from the dataset.

the mixtures of normals distributions to 2.5%.[11]

An important issue here concerns the effect of asymmetries. Consider for instance the subperiods 1941 – 45, 1976 – 80 and 1981 – 85. With Student t errors, the p-values for these subperiods are not significant since they exceed 2.5%, yet they remain below 5%. Although we emphasize the importance of accounting for the joint characteristic of our null hypothesis, this result remains empirically notable. The results of the symmetric stable errors are not substantially different from those of the elliptical distributions. This result is interesting since it is often postulated that extreme kurtosis may affect the CAPM test. However, when asymmetries are introduced, the p-values are definitively larger and not significant.

The results for the stable distribution differ in one important aspect from the case of elliptical errors. Interestingly, we have observed that the MC p-values increase almost monotonically with β_s and decrease almost monotonically with α_s (for $\beta_s > 0$ and $\alpha_s < 2$); recall that $\beta_s = 0$

[11] In this regard, we emphasize that the 2.5% level allotted to the distributional GF pre-test should not be perceived as an efficiency loss. From an empirical perspective, considering a distribution which is not supported by the data is clearly uninteresting; consequently, disregarding the joint characteristic of the null hypothesis (beside the fact that it is a statistical error) causes flawed and misleading decisions.

Table 6.4. Supremum p-values for various negative skewness measures

β_s	0	-1	$-.3$	$-.5$	$-.7$	$-.9$	$-.99$
1927 – 30	.367	.363	.539	.758	.830	.929	.941
1931 – 35	.298	.330	.517	.761	.906	.918	.925
1936 – 40	.316	.320	.408	.563	.651	.764	.740
1941 – 45	.031	.340	.039	.077	.152	.233	.285
1946 – 50	.002	.002	.002	.006	.026	.050	.082
1951 – 55	.001	.001	.002	.009	.028	.048	.038
1956 – 60	.001	.002	.001	.002	.002	.014	.010
1961 – 65	.001	.002	.002	.002	.004	.012	.020
1966 – 70	.001	.002	.002	.008	.014	.032	.044
1971 – 75	.011	.110	.146	.257	.382	.545	.596
1976 – 80	.017	.017	.024	.073	.149	.281	.329
1981 – 85	.023	.025	.033	.079	.128	.309	.285
1986 – 90	.004	.004	.005	.014	.020	.043	.058
1991 – 95	.287	.283	.297	.346	.355	.405	.477

Note. Numbers shown are p-values associated with our efficiency test using an MMC confidence set based method imposing asymmetric stable errors, which yields, given the specific $\beta_s < 0$, the largest MC p-value for all α_s within the specified confidence sets. The latter are reported in Table 6.2. October 1987 and January returns are excluded from the dataset.

and $\alpha_s = 2$ lead to the Gaussian distribution. In other words, the MC test is *less likely to reject* the no-abnormal returns null hypothesis *the more pronounced skewness and kurtosis* are modelled into the underlying regression errors. Furthermore, quite regularly, throughout our data set, the maximal p-value corresponds to the error distribution whose parameters are the smallest α_s and the largest β_s not rejected by our GF test. This monotonicity with respect to nuisance parameters (which we did not observe under elliptical errors) is notable. Of course, it also emphasizes the importance of our two-step test procedures which allows to rule out the values of α_s and β_s not supported by the data.

A simulation study conducted on the power of these GF tests (not reported here, but available from the authors upon request) reveals that while α_s is well estimated, the precision of the estimation of β_s raises further challenges. To the best of our knowledge however, the inference procedures we apply in this chapter are the only exact ones available to date. Here we show that the difficulty in estimating the skewness parameter has crucial implications for asset pricing tests. This result provides motivation to pursue research on exact approaches to the estimation of stable laws.

5. Conclusion

In this chapter, we have proposed likelihood based exact asset-pricing tests allowing for high-dimensional non-Gaussian and nonregular distributional frameworks. We specifically illustrate how to deal in finite samples with elliptical and stable errors with possibly unknown parameters. The tests suggested were applied to an efficiency problem in a standard asset pricing model framework with CRSP data.

Our empirical analysis reveals that abnormal returns are less prevalent when skewness is empirically allowed for; in addition, the effects of extreme kurtosis in the errors on test p-values are less marked than the effects of skewness. We view these results as a motivation for assessing the skewness corrected versions of the CAPM (introduced by Kraus and Litzenberger, 1976, among others). The regression model with stable errors provides an initial framework to assess asset pricing anomalies by modelling skewness via *unobservables*. Other skewness-justified approaches include: (i) extra pricing factors (see Fama and French, 1993, 1995; Harvey and Siddique, 2000) added to the regression, or (ii) the two-factor regression model of Barone-Adesi (1985) and Barone-Adesi et al. (2004a,b). To the best of our knowledge, the three-moments CAPM has been tested with procedures which are only asymptotically valid, even under normality. Our framework easily allows one to deal with multi-factor models; however, Barone-Adesi's (1985) model and its recent modification analyzed by Barone-Adesi et al. (2004a), Barone-Adesi et al. (2004b) impose nonlinear constraints. The latter empirical tests have not been reconsidered to date with reliable finite sample techniques. The development of exact versions of these tests and of alternative versions which correct for skewness is an appealing idea for future research.

References

Adler, R., Feldman, R., and Taqqu, M.S., eds. (2000). *A Practical Guide to Heavy Tails: Statistical Techniques for Analyzing Heavy Tailed Distributions*. Birkaüser.

Allingham, M. (1991). Existence theorems in the capital asset pricing model. *Econometrica*, 59:1169 – 1174.

Anderson, T.W. (1984). *An Introduction to Multivariate Statistical Analysis*, 2nd edition. John Wiley & Sons.

Baillie, R. and Bollerslev, T. (1989). The message in daily exchange rates: A conditional variance tale. *Journal of Business and Economic Statistics*, 7:297 – 305.

Barnard, G.A. (1963). Comment on "The spectral analysis of point processes" by M.S. Bartlett. *Journal of the Royal Statistical Society, Se-*

ries B, 25:294.

Barone-Adesi, G. (1985). Arbitrage equilibrium with skewed asset returns, *Journal of Financial and Quantitative Analysis*, 20(3):299 – 313.

Barone-Adesi, G., Gagliardini, P., and Urga, G. (2004a). A test of the homogeneity hypothesis on asset pricing models. In: E. Jurczenkoand B. Maillett (eds.), *Multi-Moment CAPM and Related Topics*. Springer-Verlag.

Barone-Adesi, G., Gagliardini, P., and Urga, G. (2004b). Testing asset pricing models with coskewness. Forthcoming in *Journal of Business and Economic Statistics*.

Beaulieu, M.-C. (1998). Time to maturity in the basis of stock market indices: Evidence from the S&P 500 and the MMI. *Journal of Empirical Finance*, 5:177 – 195.

Beaulieu, M.-C., Dufour, J.-M., and Khalaf, L. (2004). *Multivariate Tests of Mean-Variance Efficiency with Possibly Non-Gaussian Errors: An Exact Simulation-Based Approach*. Working paper, Département d'économique, Université Laval, and CIREQ, Université de Montréal.

Berk, J.B. (1997). Necessary conditions for the CAPM. *Journal of Economic Theory*, 73:245 – 257.

Breeden, D.T., Gibbons, M., and Litzenberger, R.H. (1989). Empirical tests of the consumption based CAPM. *Journal of Finance*, 44;231 – 262.

Cambanis, S., Samorodnitsky, G., and Taqqu, M.S., eds. (1991). *Stable Processes and Related Topics: A Selection of Papers from MSI Workshop*, Birkhaüser.

Campbell, J.Y., Lo, A.W., and MacKinlay, A.C. (1997). *The Econometrics of Financial Markets*. Princeton University Press.

Chamberlain, G. (1983). A characterization of the distributions that imply mean-variance utility functions. *Journal of Economic Theory*, 29:185 – 201.

Chambers, J.M., Mallows, C.L., and Stuck, B.W. (1976). A method for simulating stable random variables. *Journal of the American Statistical Association*, 71:340 – 344. Correction, 82:704, 83:581.

Dachraoui, K. and Dionne, G. (2003). *Conditions Ensuring the Separability of Asset Demand by all Risk Averse Investors*. Working paper, Risk Management Chair, HEC Montréal.

Dufour, J.-M. (1990). Exact tests and confidence sets in linear regressions with autocorrelated errors. *Econometrica*, 58:475 – 494.

Dufour, J.-M. (1997). Some impossibility theorems in econometrics, with applications to structural and dynamic models. *Econometrica*, 65:1365 – 1389.

Dufour, J.-M. (2002). Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics. Forthcoming in *Journal of Econometrics*.

Dufour, J.-M. (2003). Identification, weak instruments and statistical inference in econometrics. *Canadian Journal of Economics*, 36(4):767 – 808.

Dufour, J.-M., Farhat, A., Gardiol, L., and Khalaf, L. (1998a). Simulation-based finite sample normality tests in linear regressions. *The Econometrics Journal*, 1:154 – 173.

Dufour, J.-M., Hallin, M., and Mizera, I. (1998b). Generalized runs tests for heteroskedastic time series. *Journal of Nonparametric Statistics*, 9:39 – 86.

Dufour, J.-M. and Khalaf, L. (2001). Monte Carlo test methods in econometrics. In: B. Baltagi (ed.), *Companion to Theoretical Econometrics*, pp. 494 – 519. Blackwell Companions to Contemporary Economics, Basil Blackwell.

Dufour, J.-M. and Khalaf, L. (2002a). Exact tests for contemporaneous correlation of disturbances in seemingly unrelated regressions. *Journal of Econometrics*, 106(1):143 – 170.

Dufour, J.-M. and Khalaf, L. (2002b). Simulation based finite and large sample tests in multivariate regressions. *Journal of Econometrics*, 111(2):303 – 322.

Dufour, J.-M. and Khalaf, L. (2003). Finite sample tests in seemingly unrelated regressions, In: D.E.A. Giles (ed.), *Computer-Aided Econometrics*, pp. 11 – 35. Marcel Dekker.

Dufour, J.-M., Khalaf, L. and Beaulieu, M.-C. (2003). Exact skewness-kurtosis tests for multivariate normality and goodness-of-fit in multivariate regressions with application to asset pricing models. *Oxford Bulletin of Economics and Statistics*, 65:891 – 906.

Dufour, J.-M., Khalaf, L., Bernard, J.-T., and Genest, I. (2004). Simulation-based finite-sample tests for heteroskedasticity and ARCH effects. *Journal of Econometrics*, 122(2):317 – 347.

Dufour, J.-M. and Kiviet, J.F. (1996). Exact tests for structural change in first-order dynamic models. *Journal of Econometrics* 70:39 – 68.

Dufour, J.-M. and Kiviet, J.F. (1998). Exact inference methods for first-order autoregressive distributed lag models. *Econometrica*, 66:79 – 104.

Dwass, M. (1957). Modified randomization tests for nonparametric hypotheses. *Annals of Mathematical Statistics*, 28:181 – 187.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag.

Fama, E.F. (1965). The behaviour of stock prices. *Journal of Business*, 60:401 – 424.

Fama, E.F. and French, K.R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, 33:3 – 56.

Fama, E.F. and French, K.R. (1995). Size and book-to-market factors in earnings and returns. *The Journal of Finance*, 50:131 – 155.

Fama, E.F. and French, K.R. (2003). *The CAPM: Theory and Evidence.* Working paper, Center for Research in Security Prices, The University of Chicago,and Amos Tuck School of Business, Dartmouth College.

Gibbons, M.R., Ross, S.A., and Shanken, J. (1989). A test of the efficiency of a given portfolio. *Econometrica*, 57:1121 – 1152.

Harvey, C.R. and Siddique, A. (2000). Conditional skewness in asset pricing tests. *The Journal of Finance*, 55:1263 – 1295.

Hodgson, D.J., Linton, O., and Vorkink, K. (2002). Testing the capital asset pricing model efficiently under elliptical symmetry: A semiparametric approach. *Journal of Applied Econometrics*, 17;617 – 639.

Hodgson, D.J. and Vorkink, K. (2003). Efficient estimation of conditional asset pricing models. *Journal of Business and Economic Statistics*, 21:269 – 283.

Ibragimov, I.A. and Linnik, Y.V. (1975). *Independent and Stationary Sequences of Random Variables.* Wolters-Nordhoff.

Ingersoll, J. (1987). *Theory of Financial Decision Making.* Rowman & Littlefield.

Jobson, J.D. and Korkie, B. (1989). A performance interpretation of multivariate tests of asset set intersection, spanning, and mean-variance efficiency. *Journal of Financial and Quantitative Analysis*, 24:185 – 204.

Kiviet, J.F. and Dufour, J.-M. (1997). Exact tests in single equation autoregressive distributed lag models. *Journal of Econometrics*, 80:325 – 353.

Kraus, A. and Litzenberger, R. (1976). Skewness preference and the valuation of risk assets. *Journal of Finance*, 31:1085 – 1100.

Lehmann, E.L. (1986). *Testing Statistical Hypotheses*, 2nd edition. John Wiley & Sons.

Li, H. and Maddala, G.S. (1996). Bootstrapping time series models. *Econometric Reviews*, 15:115 – 158.

MacKinlay, A.C. (1987). On multivariate tests of the Capital Asset Pricing Model. *Journal of Financial Economics*, 18:341 – 372.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36:394 – 419.

Mardia, K.V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika*, 57:519 – 530.

Meerschaert, M.M. and Scheffler, H.-P. (2001). *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and*

Practice. John Wiley & Sons.

Mittnik, S., Paolella, M. and Rachev, S. (2000). Stable paretian modeling in finance: Some empirical and theoretical aspects. In Adler et al. (2000).

Nielsen, L.T. (1990). Existence of equilibrium in CAPM. *Journal of Economic Theory* 52:223 – 231.

Rachev, S., Kim, J.-R., and Mittnik, S. (1999a). Stable paretian econometrics, Part I. *The Mathematical Scientist* 24:24 – 55.

Rachev, S., Kim, J.-R., and Mittnik, S. (1999b). Stable paretian econometrics. Part II. *The Mathematical Scientist*, 24:113 – 127.

Rachev, S. and Mittnik, S. (2000). *Stable Paretian Models in Finance*, John Wiley & Sons.

Rao, C.R. (1973). *Linear Statistical Inference and its Applications*, 2nd edition. John Wiley & Sons.

Ross, S.A. (1978). Mutual fund separation in financial theory — The separating distributions. *Journal of Economic Theory*, 17:254 – 286.

Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman & Hall.

Samuelson, P. (1967). On multivariate tests of the Capital Asset Pricing Model. *Journal of Financial and Quantitative Analysis*, 2:107 – 122.

Shanken, J. (1996). Statistical methods in tests of portfolio efficiency: A synthesis. In: G.S. Maddala and C.R. Rao (eds.), *Handbook of Statistics 14: Statistical Methods in Finance*, pp. 693 – 711. North-Holland.

Stewart, K.G. (1997). Exact testing in multivariate regression. *Econometric Reviews*, 16:321 – 352.

Uchaikin, V.V. and Zolotarev, V.M. (1999). *Chance and Stability. Stable Distributions and Their Applications*. VSP, Zeist.

Vorkink, K. (2003). Return distributions and improved tests of asset pricing models. *The Review of Financial Studies*, 16(3):845 – 874.

Weron, R. (1996). On the Chambers – Mallows – Stuck method for simulating skewed stable random variables. *Statistics and Probability Letters*, 28:165 – 171.

Zolotarev, V.M. (1986). *One-Dimensional Stable Distributions*, Vol. 65 of *Translations of Mathematical Monographs*, American Mathematical Society.