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NONLINEAR HYPOTHESES, INEQUALITY RESTRICTIONS, AND NON-NESTED HYPOTHESES: EXACT SIMULTANEOUS TESTS IN LINEAR REGRESSIONS

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In the context of the classical linear model, the problem of comparing two arbitrary hypotheses on the regression coefficients is considered. Problems involving nonlinear hypotheses, inequality restrictions, or non-nested hypotheses are included as special cases. Exact bounds on the null distribution of likelihood ratio statistics are derived. The bounds are based on the central Fisher distribution and are very easy to use. In an important special case, a bounds test similar to the Durbin-Watson test is proposed. Multiple testing problems are also studied: the bounds obtained for a single pair of hypotheses are shown to enjoy a simultaneity property that allows one to combine any number of tests. This result extends to nonlinear hypotheses a well-known result given by Scheffé for linear hypotheses. A method of building bounds induced tests is also suggested.

KEYWORDS: Linear regression, nonlinear hypotheses, inequality restrictions, non-nested hypotheses, multiple testing, simultaneous inference, exact tests, bounds tests.

1. INTRODUCTION

IN THIS PAPER, we study the problem of testing completely arbitrary restrictions on the coefficients of a standard linear regression. Let

$$y = X\beta + u, \quad u \sim N\left[0, \sigma^2 I_n\right], \qquad \text{(Assumption L)}$$

where y is an $n \times 1$ vector of observations on a dependent variable, X is an $n \times k$ fixed matrix with $1 \leqslant \text{rank}(X) = k < n$, β is a $k \times 1$ vector of unknown coefficients, u is an $n \times 1$ vector of random disturbances with unknown variance $\sigma^2 > 0$. Let C be a $q \times k$ matrix of rank q $(1 \leqslant q \leqslant k)$, Γ_0 a nonempty subset of $\mathbb{R}^q$, and Ω_1 a nonempty subset of $\mathbb{R}^k$. We consider the problem of testing $H_0: C\beta \in \Gamma_0$ against $H_1: \beta \in \Omega_1$. We also examine the related multiple testing problems.

Many models in econometrics contain nonlinear restrictions, such as $\beta_1\beta_2 = 1$ and, more generally, $g(C\beta) = 0$ or $h(\beta) = 0$, where $g(\cdot)$ and $h(\cdot)$ are vector-valued functions. Economic theory often suggests inequality restrictions, such as $g(C\beta) \geqslant 0$ or $h(\beta) \geqslant 0$ (for example, sign, monotonicity, convexity, positive-definiteness assumptions), or leads to comparisons between non-nested hypotheses. The general setup described above includes as special cases all possible hypothesis pairs about β. Note that $\Omega_1 = \mathbb{R}^k$ corresponds to the case where β is unrestricted under the alternative hypothesis. With $q = k$ and $C = I_k$, the null

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hypothesis can take any form. Clearly, H_0 and H_1 as given above can be non-nested.

Let

$$(1.1)\qquad S(\bar{\beta}) = \|y - X\bar{\beta}\|^2 = (y - X\bar{\beta})'(y - X\bar{\beta}), \quad \bar{\beta} \in \mathbb{R}^k,$$

$$(1.2)\qquad SS(A) = \text{Inf}\{S(\bar{\beta}) : \bar{\beta} \in A\} = \inf_{\bar{\beta} \in A} S(\bar{\beta}), \quad \varnothing \neq A \subseteq \mathbb{R}^k.$$ [2]

A widely applicable method for testing hypotheses on β is to reject the null hypothesis when the logarithm of the likelihood ratio (LR) is too large. The critical region for testing H_0 against H_1 is then $R > c(\alpha)$, where

$$(1.3)\qquad R = (n/2)\log(SS_0/SS_1),$$

$SS_i = SS(\Omega_i)$, $i = 0, 1$, $\Omega_0 = \{\bar{\beta} \in \mathbb{R}^k : C\bar{\beta} \in \Gamma_0\}$, and $c(\alpha)$ depends on the level α of the test. Since $(SS_0 - SS_1)/SS_1 = \exp(2R/n) - 1$, an equivalent critical region is given by $(SS_0 - SS_1)/SS_1 > d(\alpha)$, where $d(\alpha) = \exp[2c(\alpha)/n] - 1$. A basic problem here is to determine the critical value that yields a test with the desired level α.[3]

When the null hypothesis is a set of linear restrictions on β, i.e. $\Omega_0 = \{\bar{\beta} \in \mathbb{R}^k : C\bar{\beta} = \gamma_0\}$ for some $\gamma_0 \in \mathbb{R}^q$ (*linear hypothesis*), and when $\Omega_1 = \mathbb{R}^k$, it is well known that

$$(1.4)\qquad P\left[(SS_0 - SS_1)/SS_1 \geqslant qF_\alpha(q, n-k)/(n-k)\right] = \alpha$$

when $\beta \in \Omega_0$; $F_\alpha(\cdot)$ is defined by

$$(1.5)\qquad P\left[F(\nu_1, \nu_2) \geqslant F_\alpha(\nu_1, \nu_2)\right] = \alpha, \quad 0 \leqslant \alpha \leqslant 1,$$

where $F(\nu_1, \nu_2)$ follows a central Fisher distribution with (ν_1, ν_2) degrees of freedom (we set $F_0(\nu_1, \nu_2) = +\infty$ and $F_1(\nu_1, \nu_2) = 0$). On the other hand, when the problem does not have this form (or the form of two nested linear hypotheses), it is much more difficult to find an exact critical value. We shall call such problems *nonregular problems*. Tests of nonlinear hypotheses such as $h(\beta) = 0$ are typically based on asymptotic chi-square approximations (see Amemiya (1983), Judge et al. (1985, Ch. 6), and Malinvaud (1981, Ch. 9)). Note that various regularity conditions on the function $h(\cdot)$ are needed for the asymptotic theory to apply. When inequality restrictions (linear or nonlinear) are involved, the astymptotic distribution is not generally chi-square; it can take an appreciably more complex form, e.g., a mixture of chi-square distributions (see Gouriéroux,

[2] In this paper, β refers to the true coefficient vector while $\bar{\beta}$ refers to any possible value in $\mathbb{R}^k$. We use the infimum operator (Inf) rather than the minimum operator (Min) because no special form is imposed on the hypotheses: while the infimum is always defined (the function $S(\bar{\beta})$ is bounded downward), a minimum may not exist (e.g., in certain situations where A is open). Of course, when a minimum does exist, it is equal to the infimum.

[3] Note that, without additional conditions on the form of the hypotheses, alternative testing methods, like the Wald or the Lagrange multiplier method, may be difficult to implement. For example, a Wald test could be based on the difference $\hat{\beta}_0 - \hat{\beta}_1$, where $\hat{\beta}_i$ is a value giving the minimum of SS_i, but the asymptotic covariance matrix and distribution of $\hat{\beta}_0 - \hat{\beta}_1$ may be difficult to derive without further regularity conditions.

Holly, and Monfort (1982)). In nonregular problems, algorithms to compute exact critical values for LR statistics have been developed only in a few special cases: problems with linear inequalities (see Farebrother (1986), Judge and Yancey (1986), and Wolak (1987)) and comparisons between two linear non-nested models (King (1985)). Even in these cases, the critical value depends on the regressor matrix and a fair amount of computation may be required to find the required critical value. No finite-sample results seem available for more complex problems.

An important related difficulty is the problem of multiple testing. It is common practice in statistical and econometric analysis to perform several tests on the parameters of a model, either jointly or sequentially. This raises the issue of controlling overall significance levels, and leads to so-called "induced tests" and multiple comparison procedures. For linear hypotheses in the classical linear model, several methods are available for multiple hypothesis testing; see Miller (1966, 1977) and Savin (1980, 1984). However, when nonlinear hypotheses, inequality restrictions, or non-nested hypotheses are involved, no exact results seem available on multiple testing.

In this paper, we present a number of finite-sample results on the marginal and joint distributions of likelihood ratio statistics for nonregular problems in the classical linear model. We study in turn the problem of testing a single pair of hypotheses, $H_0: C\beta \in \Gamma_0$ against $H_1: \beta \in \Omega_1$, and the corresponding multiple testing problems. For the single hypothesis pair problem, we show that, under H_0, the inequality

$$(1.6) \qquad P\left[(SS_0 - SS_1)/SS_1 \geqslant qF_\alpha(q, n-k)/(n-k)\right] \leqslant \alpha$$

holds *exactly, irrespective of the forms of the null and alternative hypotheses* ($0 \leqslant \alpha \leqslant 1$). Consequently, the critical value $Q_\alpha \equiv qF_\alpha(q, n-k)/(n-k)$ is *conservative* (at level α) whenever the null hypothesis can be written in the form $C\beta \in \Gamma_0$, where $\varnothing \neq \Gamma_0 \subseteq \mathbb{R}^q$; there is no restriction on the type of the alternative $\beta \in \Omega_1$. Q_α may be viewed as an upper bound on the (usually unknown) critical value $\bar{Q}_\alpha$ that yields a test of *size* exactly α (we call $\bar{Q}_\alpha$ a *tight critical value*). Though the proof of this result is remarkably simple, it does not seem to be known at all. Second, we show that the inequality (1.6) still holds if $SS_1 \equiv SS(\Omega_1)$ is replaced by $S(\tilde{\beta})$, where $\tilde{\beta}$ is *any estimator* of β constrained by the *alternative* hypothesis ($\tilde{\beta} \in \Omega_1$). The estimator $\tilde{\beta}$ need not be consistent. This property can be helpful when it is difficult to compute the minimum sum of squares under the alternative. Third, we give a general sufficient condition under which one can easily find a point Q'_α such that

$$(1.7) \qquad P\left[(SS_0 - SS_1)/SS_1 > Q'_\alpha\right] \geqslant \alpha$$

for all distributions in H_0. The latter can be viewed as a *liberal* critical value, which gives a lower bound for $\bar{Q}_\alpha$. For example, the condition obtains whenever H_0 is nested in H_1 and the null hypothesis imposes some linear restrictions on β (plus possibly a number of nonlinear constraints). When Q'_α is available, one can combine it with Q_α and build an exact bounds test similar to the Durbin-Watson

(1950) test. Fourth, we discuss the application of these results to problems with linear inequalities and comparisons between two non-nested linear models. Given that our bounds are very easy to compute, while the search for tight critical values may require expensive calculations, the bounds can play a useful role even in these relatively simple cases. We also discuss how to apply the results to nonlinear problems.

With regard to multiple testing problems, we consider a collection of hypothesis pairs $\{(\Gamma_{a0}, \Omega_{a1}): a \in J,\ \Gamma_{a0} \subseteq \mathbb{R}^q,\ \Omega_{a1} \subseteq \mathbb{R}^k\}$ and the associated LR tests. The index set J may be finite or infinite, fixed or stochastic. For this case, we have the inequality

$$(1.8) \qquad P\left[\sup_{a \in J} \{(SS_{a0} - SS_{a1})/SS_{a1}\} \geqslant qF_\alpha(q, n-k)/(n-k)\right] \leqslant \alpha$$

provided $C\beta \in \bigcap_{a \in J} \Gamma_{a0}$ (with probability 1).[4] Thus, if $\Gamma_0 \subseteq \bigcap_{a \in J} \Gamma_{a0}$ (with probability 1), the induced test that rejects $C\beta \in \Gamma_0$ when $(SS_{a0} - SS_{a1})/SS_{a1} > Q_\alpha$ for some $a \in J$ has size not greater than α. This result includes as a special case an important implication of the well-known result due to Scheffé (1953, 1959) for linear hypotheses. We find that the conservative simultaneous critical bounds given by Scheffé for linear hypotheses on a vector $\gamma \equiv C\beta$ retain these characteristics when applied to any collection of hypotheses on γ. Second, we observe that (1.8) still holds if SS_{a1} is replaced by $S(\tilde{\beta}_a)$ where $\tilde{\beta}_a$ is *any* estimator of β under the *alternative hypothesis* $\beta \in \Omega_{a1}$. Third, we give a sufficient condition under which a liberal bound, like the one in (1.7), can be obtained for a family of tests. Fourth, when the latter is applicable, we describe a simple and intuitive way of building *bounds induced tests*.

In Section 2, we derive two general theorems from which the other results follow and we study in detail the problem of testing a single pair of hypotheses. Section 3 examines a number of special cases of these results. Section 4 discusses multiple testing problems. Section 5, finally, summarizes the main results and concludes.

2. TESTS OF NONLINEAR HYPOTHESES

In this section, we study the problem of testing $C\beta \in \Gamma_0$ against $\beta \in \Omega_1$, where $\Gamma_0 \subseteq \mathbb{R}^q$ and $\Omega_1 \subseteq \mathbb{R}^k$. This includes as a special case the problem of testing $C\beta \in \Gamma_0$ against $C\beta \in \Gamma_1$, where Γ_0 and Γ_1 are two arbitrary subsets of $\mathbb{R}^q$.

To do this, we shall exploit the following general idea. Consider a family of distributions indexed by the parameter vector $\boldsymbol{\theta}$ in some space ω, and a real-valued statistic $T(\omega_0, \omega_1)$ for testing $H_0: \boldsymbol{\theta} \in \omega_0$ against $H_1: \boldsymbol{\theta} \in \omega_1$, where $\omega_0 \subseteq \omega$ and $\omega_1 \subseteq \omega$. Further suppose that we can find another statistic $\overline{T}(\boldsymbol{\theta})$ such

[4] The expression "with probability 1" is used because the index set J may be stochastic. For example, J is stochastic when the null hypotheses $C\beta \in \Gamma_{a0}$ (i.e., the indices a included in J) are selected through a pre-testing process. When the distribution of J depends on a parameter, the condition "with probability 1" must be uniform in this parameter.

that

$$T(\omega_0, \omega_1) \leqslant \bar{T}(\theta)$$

when $\theta \in \omega_0$, and

$$P_\theta\left[\bar{T}(\theta) \geqslant x\right] = \bar{G}(x), \quad \forall\, x,$$

where the function $\bar{G}(x)$ does not depend on θ and P_θ refers to the probability measure associated with θ. Then we have

$$P_\theta\left[T(\omega_0, \omega_1) \geqslant x\right] \leqslant \bar{G}(x), \quad \forall\, x,$$

when $\theta \in \omega_0$. If the function $\bar{G}(x)$ is tractable, we can use it to bound $P_\theta[\bar{T}(\theta) \geqslant x]$ and obtain upper bounds on critical values for $T(\omega_0, \omega_1)$. Similarly, if we can find another statistic $\tilde{T}(\theta)$ such that

$$T(\omega_0, \omega_1) \geqslant \tilde{T}(\theta), \quad \text{when} \quad \theta \in \omega_0,$$

$$P_\theta\left[\tilde{T}(\theta) \geqslant x\right] = \tilde{G}(x), \quad \forall\, x,$$

where $\tilde{G}(x)$ does not depend on θ, then

$$P_\theta\left[T(\omega_0, \omega_1) \geqslant x\right] \geqslant \tilde{G}(x), \quad \forall\, x,$$

when $\theta \in \omega_0$. If $\tilde{G}(x)$ is known, we can use it to obtain lower bounds on critical values for $T(\omega_0, \omega_1)$.

It turns out that this approach can be applied fairly easily to deal with nonregular problems in the context of the standard linear model. Using it, we first prove a general theorem holding for collections of LR statistics.

Let

$$A(q, k) = \left\{(A, B) : A \subseteq \mathbb{R}^q,\ B \subseteq \mathbb{R}^k,\ A \neq \varnothing,\ B \neq \varnothing\right\} \tag{2.1}$$

represent the set of all possible comparisons of a null hypothesis on $C\beta$ with some alternative hypothesis on β. Typically, the investigator is interested by only a subset $H \subseteq A(q, k)$ of the possible comparisons (one pair, for example). We consider the family of LR statistics associated to a subset of these comparisons $H(J) = \{(\Gamma_{a0}, \Omega_{a1}) : a \in J\} \subseteq H$. The index set J may be fixed or data-dependent. Then the following theorem holds.

THEOREM 1 (Simultaneous conservative bound): *Let Assumption L hold. Let C be a $q \times k$ matrix of rank q $(1 \leqslant q \leqslant k)$, $H(J)$ and H nonempty subsets of $A(q, k)$ such that*

$$H(J) = \left\{(\Gamma_{a0}, \Omega_{a1}) : a \in J,\ \Gamma_{a0} \subseteq \mathbb{R}^q,\ \Omega_{a1} \subseteq \mathbb{R}^k\right\} \subseteq H$$

where the index set J may be stochastic, $\Omega_{a0} = \{\bar{\beta} \in \mathbb{R}^q : C\bar{\beta} \in \Gamma_{a0}\}$, $SS_{ai} = SS(\Omega_{ai})$, $i = 0, 1$, $T_a = (SS_{a0} - SS_{a1})/SS_{a1}$, and $Q_\alpha = qF_\alpha(q, n-k)/(n-k)$, $0 \leqslant \alpha \leqslant 1$. Then, if $C\beta \in \bigcap_{a \in J} \Gamma_{a0}$ (with probability 1), we have

$$P\left[T_b \geqslant Q_\alpha\right] \leqslant P\left[\operatorname{Sup}\{T_a : a \in J\} \geqslant Q_\alpha\right] \leqslant \alpha, \tag{2.2}$$

where the index b can be chosen in J by any rule.

PROOF: Let $\gamma = C\beta$, where β is the true value of the coefficient vector, $\omega_0 = \{\bar{\beta} \in \mathbb{R}^k : C\bar{\beta} = \gamma\}$, and $\overline{SS}_0 = SS(\omega_0)$. We suppose that the "null hypothesis" is true, i.e. $C\beta \in \bigcap_{a \in J} \Gamma_{a0}$ (when J is stochastic, it is assumed that this event has probability 1). Then, for all $a \in J$, we have $\gamma \in \Gamma_{a0}$, $\omega_0 \subseteq \Omega_{a0} \subseteq \mathbb{R}^k$,

$$(2.3) \qquad SS \leqslant SS_{a0} \leqslant \overline{SS}_0,$$

and

$$(2.4) \qquad SS \leqslant SS_{a1},$$

where $SS = SS(\mathbb{R}^k)$. From (2.3) and (2.4), we see that

$$(2.5) \qquad T_a = (SS_{a0} - SS_{a1})/SS_{a1} \leqslant (\overline{SS}_0 - SS)/SS$$

for all $a \in J$, hence

$$(2.6) \qquad T_b \leqslant \operatorname{Sup}\{T_a : a \in J\} \leqslant (\overline{SS}_0 - SS)/SS$$

for any $b \in J$. Since SS is the unrestricted minimum sum of squares for model L while $SS(\omega_0)$ is the minimum sum of squares under the linear constraint $C\beta = \gamma$, we have

$$(\overline{SS}_0 - SS)/SS \sim qF(q, n-k)/(n-k)$$

under the null hypothesis. Thus, for any $b \in J$ (irrespective of the way b is chosen in J),

$$P[T_b \geqslant Q_\alpha] \leqslant P[\operatorname{Sup}\{T_a : a \in J\} \geqslant Q_\alpha] \leqslant P[(\overline{SS}_0 - SS)/SS \geqslant Q_\alpha] = \alpha.$$ [5] *Q.E.D.*

This theorem has applications to simultaneous inference problems which will be discussed in Section 4. In this section, we study the important case where only one pair of hypotheses is considered. This can be done by taking $H = H(J) = \{(\Gamma_0, \Omega_1)\}$ in Theorem 1. We then get the following corollary.

COROLLARY 1 (Conservative bound): *Let Assumption L hold. Let C be a $q \times k$ matrix of rank q ($1 \leqslant q \leqslant k$), Γ_0 a nonempty subset of $\mathbb{R}^q$, Ω_1 a nonempty subset of $\mathbb{R}^k$, $\Omega_0 = \{\bar{\beta} \in \mathbb{R}^k : C\bar{\beta} \in \Gamma_0\}$ and $SS_i = SS(\Omega_i)$, $i = 0, 1$. Then, if $C\beta \in \Gamma_0$, we have*

$$(2.7) \qquad P[(SS_0 - SS_1)/SS_1 \geqslant qF_\alpha(q, n-k)/(n-k)] \leqslant \alpha$$

for all $0 \leqslant \alpha \leqslant 1$.

[5] In this paper, we do not discuss conditions under which the variables $\operatorname{Sup}\{T_a : a \in J\}$ and T_b are measurable functions of y (with respect to the profitability measure given by model L). Even if the latter variables are not measurable functions of y, one can always interpret the operator $P[\cdot]$ in the inequality (2.2) as the outer measure induced by the probability measure given by model L (where we take into account the fact that $(\overline{SS}_0 - SS)/SS$ is measurable). *A fortiori*, $P[\cdot]$ could also be interpreted as the corresponding inner measure. A similar remark applies to Corollaries 1, 1.1, and 3. In the inequality (2.13) of Theorem 2, we can interpret $P[\cdot]$ as the inner measure (or the outer measure) induced by L, and similarly for Corollaries 2, 2.1, and 4. For further discussion of inner and outer measures, see Cohn (1980, Ch. 1).

In terms of the likelihood ratio statistic for testing $H_0: C\beta \in \Gamma_0$ against $H_1: \beta \in \Omega_1$, the inequality (2.7) can be written

$$(2.8)\qquad P\left[(n/2)\log(SS_0/SS_1) \geqslant (n/2)\log(1+Q_\alpha)\right] \leqslant \alpha, \quad 0 \leqslant \alpha \leqslant 1,$$

where $Q_\alpha = qF_\alpha(q, n-k)/(n-k)$. However, it is more convenient to work with the equivalent statistic $(SS_0 - SS_1)/SS_1$. Note also that $\geqslant$ can be replaced by $>$ in (2.2) and (2.7).[6]

The critical bound Q_α has several attractions: it is exact, easy to compute, and widely applicable. By taking $\Omega_1 = \mathbb{R}^k$, we can test any hypothesis that restricts the value of a vector $C\beta$ against the unrestricted model. By taking $q = k$ and $C = I_k$, we can consider any pair of null and alternative hypotheses on β. Clearly, null hypotheses of the form $H_0: g(C\beta) = 0$, where $g(\cdot)$ is some vector-valued function (nonlinear equality restrictions), or hypotheses of the form $H_0: g(C\beta) \geqslant 0$ (nonlinear inequality restrictions) are included as special cases. Combinations of the two types of conditions may also be considered. Similarly, the alternative hypothesis H_1 may involve any set of compatible restrictions on the regression coefficients. *A fortiori* the critical bound is also applicable to null hypotheses that involve linear inequality restrictions or linear models on which we impose such restrictions.

In all cases, we can be sure that the result $(SS_0 - SS_1)/SS_1 \geqslant Q_\alpha$ is statistically significant at level α. Note that $q_1 > q_2 \geqslant 1$ implies $q_1F_\alpha(q_1, n-k) > q_2F_\alpha(q_2, n-k)$ for $0 < \alpha < 1$. Thus, one should seek to express the null hypothesis so that q is as small as possible, since this leads to the smallest critical bound Q_α.

Minimizing the sum of squares under nonlinear constraints can be costly. When a cheaper estimator is available under the alternative hypothesis, it is sometimes possible to draw a conclusion without finding the minimum sum of squares under the alternative. This possibility is pointed out by the following corollary.

COROLLARY 1.1: *Let the assumptions of Corollary* 1 *hold and let* $\tilde{\beta}$ *be any* $k \times 1$ *random vector such that* $P[\tilde{\beta} \in \Omega_1] = 1$ *irrespective of the value of* β. *Then, if* $C\beta \in \Gamma_0$, *we have*

$$(2.9)\qquad P\left[\{SS_0 - S(\tilde{\beta})\}/S(\tilde{\beta}) \geqslant Q_\alpha\right] \leqslant P\left[(SS_0 - SS_1)/SS_1 \geqslant Q_\alpha\right] \leqslant \alpha,$$

for all $0 \leqslant \alpha \leqslant 1$, *where* $Q_\alpha = qF_\alpha(q, n-k)/(n-k)$.

PROOF: The result follows from Corollary 1 and the observation that

$$(2.10)\qquad SS_0/S(\tilde{\beta}) \leqslant SS_0/SS_1 \quad \text{with probability } 1. \qquad Q.E.D.$$

[6] In this paper, we do not discuss computation methods for minimizing the sum of squares with nonlinear and/or inequality constraints. On these problems, see Avriel (1976), Bezaraa and Shetty (1979), Gill et al. (1981), Judge and Takayama (1966), Liew (1976), and Schmidt and Thomson (1982).

Q_α is thus a general upper bound on the critical values of tests based on statistics of the form $[SS_0 - S(\tilde{\beta})]/S(\tilde{\beta})$ when $\tilde{\beta}$ is any estimator of β that satisfies the alternative hypothesis $\beta \in \Omega_1$ with probability 1. Note that the "estimator" $\tilde{\beta}$ could be based on a sample different from the one used to obtain SS_0. Instead of (2.9), we can state

$$(2.11) \quad P\left[\{SS_0 - S(\tilde{\beta})\}/S(\tilde{\beta}) \leqslant x\right] \geqslant P\left[F(q, n-k) \leqslant (n-k)x/q\right], \quad \forall\, x.$$

Further, by (2.10), $\{SS_0 - S(\tilde{\beta})\}/S(\tilde{\beta}) > Q_\alpha$ implies that $(SS_0 - SS_1)/SS_1 > Q_\alpha$. A simple example of an estimator $\tilde{\beta}$ to which the above corollary applies is the case where $\tilde{\beta}$ is the estimator of β at any step of the process by which the sum of squares under H_1 is minimized iteratively (provided $\tilde{\beta} \in \Omega_1$).

The critical bound discussed in Corollary 1 is *conservative* in the sense that the test which rejects H_0 when $(SS_0 - SS_1)/SS_1$ is large is *certainly significant* at level α.[7] A critical value $\bar{Q}_\alpha$ such that $\bar{Q}_\alpha < Q_\alpha$ and

$$(2.12) \quad \operatorname*{Sup}_{\beta \in \Omega_0} P\left[(SS_0 - SS_1)/SS_1 > \bar{Q}_\alpha\right] = \alpha$$

may exist but is typically more difficult to determine. In certain cases, it is also possible to find a *liberal* critical value, i.e. a bound Q'_α such that $Q'_\alpha \leqslant \bar{Q}_\alpha$: if we observe $(SS_0 - SS_1)/SS_1 \leqslant Q'_\alpha$, we can be sure that the test is *not significant* at level α. The two following propositions describe such cases. As for the conservative bound, we first give a general proposition holding for families of LR statistics.

THEOREM 2 (Simultaneous liberal bound): *Let Assumption L hold. Let C_1 be a $p \times k$ matrix of rank p $(1 \leqslant p \leqslant k)$, $\varnothing \neq \tilde{\Omega}_i \subseteq \mathbb{R}^k (i = 0, 1)$, J an index set (possibly stochastic), $\tilde{H}(J)$ and $\tilde{H}$ nonempty subsets of $A(k, k)$ such that*

$$\tilde{H}(J) = \left\{(\Omega_{a0}, \Omega_{a1}) : a \in J,\, \Omega_{a0} \subseteq \mathbb{R}^k,\, \Omega_{a1} \subseteq \mathbb{R}^k\right\} \subseteq \tilde{H},$$

$$\Omega_{a0} \subseteq \tilde{\Omega}_0 = \left\{\bar{\beta} \in \mathbb{R}^k : C_1\bar{\beta} - \gamma_{10} = 0\right\}, \quad \forall\, a \in J,$$

$$\tilde{\Omega}_1 = \left\{\bar{\beta} \in \mathbb{R}^k : D\left(C_1\bar{\beta} - \gamma_{10}\right) = 0\right\} \subseteq \Omega_{a1}, \quad \forall\, a \in J,$$

for some vector $\gamma_{10} \in \mathbb{R}^p$ and some $p_1 \times p$ matrix D $(1 \leqslant p_1 \leqslant p)$. Let also $SS_{ai} = SS(\Omega_{ai})$, $i = 0, 1$, $T_a = (SS_{a0} - SS_{a1})/SS_{a1}$, and $Q'_\alpha = (p-d)F_\alpha(p-d, n-k+d)/(n-k+d)$, where $d = \operatorname{rank}(D)$ and $0 \leqslant \alpha \leqslant 1$. Then, if $\beta \in \bigcap_{a \in J}\Omega_{a0}$ (with probability 1), we have

$$(2.13) \quad P\left[T_b > Q'_\alpha\right] \geqslant P\left[\operatorname{Inf}\{T_a : a \in J\} > Q'_a\right] \geqslant \alpha,$$

where the index b can be chosen in J by any rule.

PROOF: Let $\tilde{SS}_i = SS(\tilde{\Omega}_i)$, $i = 0, 1$. For all $a \in J$, we have (with probability 1)

$$\Omega_{a0} \subseteq \tilde{\Omega}_0 \subseteq \tilde{\Omega}_1 \subseteq \Omega_{a1},$$

[7] For other examples of conservative tests (exact or asymptotic) based on a similar bounding argument, see Durbin and Watson (1950), Varian (1985), and Epstein and Yatchew (1985).

hence

$$SS(\Omega_{a1}) \leqslant SS(\tilde{\Omega}_1) \leqslant SS(\tilde{\Omega}_0) \leqslant SS(\Omega_{a0})$$

and

$$(2.14) \qquad (SS_{a0} - SS_{a1})/SS_{a1} \geqslant (\tilde{SS_0} - \tilde{SS_1})/\tilde{SS_1}.$$

Thus, for any $b \in J$,

$$T_b \geqslant \operatorname{Inf}\{T_a : a \in J\} \geqslant (\tilde{SS_0} - \tilde{SS_1})/\tilde{SS_1}.$$

$\tilde{SS_0}$ is the minimum sum of squares under the linear restriction $C_1\bar{\beta} - \gamma_{10} = 0$ and $\tilde{SS_1}$ is the minimum sum of squares under the weaker linear restriction $D(C_1\bar{\beta} - \gamma_{10}) = 0$. Further $\operatorname{rank}(C_1) = p$ and $\operatorname{rank}(DC_1) = \operatorname{rank}(D) \equiv d$, where $0 \leqslant d \leqslant p$. In the first case, we have p linearly independent restrictions on β while, in the second case, there are only d linearly independent restrictions. Thus, if we suppose that $\beta \in \bigcap_{a \in J}\Omega_{a0}$ with probability 1 (i.e., under the "null hypothesis"), we have

$$(\tilde{SS_0} - \tilde{SS_1})/\tilde{SS_1} \sim (p-d)F(p-d, n-k+d)/(n-k+d);$$

hence, for any b chosen in J,

$$\begin{aligned} P[T_b > Q'_\alpha] &\geqslant P[\operatorname{Inf}\{T_a : a \in J\} > Q'_\alpha] \\ &\geqslant P[(\tilde{SS_0} - \tilde{SS_1})/\tilde{SS_1} > Q'_\alpha] = \alpha. \end{aligned}$$

Q.E.D.

In the case of a single pair of hypotheses, we have $\tilde{H} = \tilde{H}(J) = \{(\Omega_0, \Omega_1)\}$ in Theorem 2, which yields the following corollary.

COROLLARY 2 (Liberal bound): *Let Assumption L hold. Let C_1 be a $p \times k$ matrix of rank p $(1 \leqslant p \leqslant k)$, Ω_0, $\tilde{\Omega}_0$, Ω_1, and $\tilde{\Omega}_1$ four nonempty subsets of $\mathbb{R}^k$ such that*

$$\Omega_0 \subseteq \tilde{\Omega}_0 = \{\bar{\beta} \in \mathbb{R}^k : C_1\bar{\beta} - \gamma_{10} = 0\},$$

$$\tilde{\Omega}_1 = \{\bar{\beta} \in \mathbb{R}^k : D(C_1\bar{\beta} - \gamma_{10}) = 0\} \subseteq \Omega_1,$$

for some vector $\gamma_{10} \in \mathbb{R}^p$ and some $p_1 \times p$ matrix D $(1 \leqslant p_1 \leqslant p)$, and let $SS_i = SS(\Omega_i)$, $i = 0, 1$. Then, if $\beta \in \Omega_0$, we have

$$(2.15) \qquad P[(SS_0 - SS_1)/SS_1 > (p-d)F_\alpha(p-d, n-k+d)/(n-k+d)] \geqslant \alpha$$

for all $0 \leqslant \alpha \leqslant 1$, where $d = \operatorname{rank}(D)$.

Note that $>$ may be replaced by $\geqslant$ in (2.13) and (2.15). A corollary analogous to Corollary 1.1 also holds for Corollary 2. One finds that the liberal critical bound given in the latter proposition remains applicable when SS_0 is replaced by $S(\tilde{\beta})$, where $\tilde{\beta}$ is any estimator of β under the null hypothesis (rather than the alternative hypothesis, as assumed in Corollary 1.1).

COROLLARY 2.1: *Let the assumptions of Corollary 2 hold and let $\tilde{\beta}$ be any $k \times 1$ random vector such that $P[\tilde{\beta} \in \Omega_0] = 1$ irrespective of the value of β. Then, if*

$\beta \in \Omega_0$, we have

$$(2.16) \qquad P\left[\left\{S(\tilde{\beta}) - SS_1\right\}/SS_1 > Q'_\alpha\right] \geqslant P\left[(SS_0 - SS_1)/SS_1 > Q'_\alpha\right] \geqslant \alpha$$

for all $0 \leqslant \alpha \leqslant 1$, where $Q'_\alpha = (p-d)F_\alpha(p-d, n-k+d)/(n-k+d)$ and $d =$ rank (D).

PROOF: The result follows from Corollary 2 and the observation that

$$S(\tilde{\beta})/SS_1 \geqslant SS_0/SS_1 \quad \text{with probability 1.} \qquad Q.E.D.$$

Clearly, the critical bound given by Corollary 2 is not always applicable. In particular, the two hypotheses must be nested ($\Omega_0 \subseteq \Omega_1$). An important case where it can be used is the one where the null hypothesis imposes a number of linear restrictions on β (plus possibly some nonlinear constraints), while the alternative hypothesis relaxes all or some of these linear restrictions (and may allow other possibilities). For example, let C_1 and $C = [C_1', C_2']'$ be $p \times k$ and $q \times k$ full row rank matrices ($1 \leqslant p \leqslant q \leqslant k$), with $C = C_1$ when $p = q$, and let

$$(2.17) \qquad \Omega_0 = \left\{\bar{\beta} \in \mathbb{R}^k : C_1\bar{\beta} - \gamma_{10} = 0 \text{ and } C\bar{\beta} \in \Gamma_0\right\},$$

$$\Omega_1 = \left\{\bar{\beta} \in \mathbb{R}^k : D\left(C_1\bar{\beta} - \gamma_{10}\right) = 0 \text{ or } \bar{\beta} \in \omega_1\right\},$$

where $\Gamma_0 \subseteq \mathbb{R}^q$ and $\omega_1 \subseteq \mathbb{R}^k$ such that $\Omega_0 \neq \varnothing$. Then the conditions of Corollaries 1 and 2 are satisfied and we have for $\beta \in \Omega_0$

$$(2.18) \qquad P\left[(SS_0 - SS_1)/SS_1 > Q_\alpha\right] \leqslant \alpha \leqslant P\left[(SS_0 - SS_1)/SS_1 > Q'_\alpha\right],$$

where $Q_\alpha = qF_\alpha(q, n-k)/(n-k)$ and $Q'_\alpha = (p-d)F_\alpha(p-d, n-k+d)/(n-k+d)$. It is easy to see that $Q'_\alpha \leqslant Q_\alpha$. Let $d < p$ ($d = p$ implies $Q'_\alpha = 0$). We test $H_0 : \beta \in \Omega_0$ against $H_1 : \beta \in \Omega_1$ by rejecting H_0 when $(SS_0 - SS_1)/SS_1$ is greater than a given constant. If $(SS_0 - SS_1)/SS_1 > Q_\alpha$, we can be sure that the test is significant at level α. If $(SS_0 - SS_1)/SS_1 \leqslant Q'_\alpha$, we can be sure that the test is not significant at level α. Otherwise, the test may be viewed as inconclusive. This suggests a bounds procedure similar to the one suggested by Durbin and Watson (1950) for testing the autocorrelation of disturbances:

(2.19) reject H_0 if $(SS_0 - SS_1)/SS_1 > Q_\alpha$,
accept H_0 if $(SS_0 - SS_1)/SS_1 \leqslant Q'_\alpha$,
test inconclusive, otherwise.

If $D = 0$, we have $\Omega_1 = \mathbb{R}^k$: SS_1 is the unrestricted sum of squares. When $d = 0$ and $C_1 = C$, we have $p = q$, $d = 0$, $\Omega_0 = \{\bar{\beta} \in \mathbb{R}^k : C_1\bar{\beta} - \gamma_{10} = 0\}$, $\Omega_1 = \mathbb{R}^k$, and $Q_\alpha = Q'_\alpha : (SS_0 - SS_1)/SS_1$ follows a $qF(p-d, n-k+d)/(n-k+d)$ distribution under the null hypothesis and the bounds procedure becomes equivalent to the usual F test. Table I gives some numerical values of the bounds Q_α and Q'_α. Further, it is straightforward to see that $Q_\alpha - Q'_\alpha$ tends to zero as $n \to \infty$ (with k, q, p, and d fixed). Thus, if $\bar{Q}_\alpha$ is the tight critical value given by (2.12) and

$$\lim_{n \to \infty} P\left[(SS_0 - SS_1)/SS_1 > \bar{Q}_\alpha\right] = 1 \quad \text{when} \quad \beta \notin \Omega_0$$

TABLE I
EXAMPLES OF CRITICAL BOUNDS ($\alpha = .05$)

n	k	q	p	d	Q'_α	Q_α
30	3	2	1	0	.16	.25
	4	2	1	0	.16	.26
	4	3	2	1	.16	.34
	5	2	1	0	.17	.27
	5	3	2	0	.27	.36
	5	3	2	1	.16	.36
60	3	2	1	0	.07	.11
	4	2	1	0	.07	.11
	4	3	2	1	.07	.15
	5	2	1	0	.07	.12
	5	3	2	0	.12	.15
	5	3	2	1	.07	.15

for *all* $0 < \alpha < 1$, we have

$$\lim_{n \to \infty} P\left[(SS_0 - SS_1)/SS_1 > Q_\alpha\right] = 1 \quad \text{when} \quad \beta \notin \Omega_0$$

(this follows from the inequality $Q'_\alpha \leqslant \bar{Q}_\alpha \leqslant Q_\alpha$ and by observing that $Q'_\alpha \to \infty$ when $\alpha \to 0$). In other words, the bounds procedure is consistent for $\beta \notin \Omega_0$ provided the corresponding test based on a tight critical value is consistent.

Note that the basic characteristic of a bounds test is that it satisfies the conditions

$$(2.20) \qquad P[\text{Rejecting } H_0 | H_0] \leqslant \alpha, \quad P[\text{Accepting } H_0 | H_0] \leqslant 1 - \alpha.$$

Rejection with a small value of α may be considered as a "strong rejection" while acceptance with α large may be viewed as a "strong acceptance." For further discussion of bounds procedures, see Dufour (1986).

3. SPECIAL CASES

To illustrate how the above results may be used, let $k = 5$ in model L and suppose we wish to test $H_0: \beta_2\beta_3 = 1$ against the unrestricted model. Then we can set $q = 2$ in Corollary 1 (because this nonlinear restriction involves two coefficients only) and we have under H_0

$$P\left[(SS_0 - SS_1)/SS_1 \geqslant 2F_\alpha(2, n-5)/(n-5)\right] \leqslant \alpha.$$

We thus get easily an exact conservative critical value for the statistic $(SS_0 - SS_1)/SS_1$. If instead we test $\bar{H}_0: \beta_2\beta_3 \leqslant 1$ against the unrestricted model, we can use the same critical value (with SS_0 obtained under the restriction $\beta_2\beta_3 \leqslant 1$). Suppose now that we wish to test $\beta_4 = 0$ when $\beta_2\beta_3 = 1$ is a *maintained* restriction. This is equivalent to testing H_0^*: $\beta_4 = 0$ and $\beta_2\beta_3 = 1$, against $H_1^*: -\infty < \beta_4 < +\infty$ and $\beta_2\beta_3 = 1$ (more simply, $H_1^*: \beta_2\beta_3 = 1$). Then we have $q = 3$ in Corollary 1 and we can also apply Corollary 2 with $p = 1$ and $d = 0$. We thus get the conservative and liberal critical values $Q_\alpha = 3F_\alpha(3, n-5)/(n-5)$ and $Q'_\alpha = F_\alpha(1, n-5)/(n-5)$. If we replace $\beta_2\beta_3 = 1$ by $\beta_2\beta_3 \leqslant 1$ in H_0^* and

H_1^*, we can use the same critical values (with SS_0 and SS_1 obtained under the inequality restriction $\beta_2\beta_3 \leqslant 1$).

To take a more complex example, consider the equation

$$y_i = \beta_0 + \sum_{j=1}^{2} \gamma_j x_{ij} + \delta_{11} x_{i1}^2 + \delta_{12}(2x_{i1}x_{i2}) + \delta_{22} x_{i2}^2 + u_i \quad (i = 1, \dots, n).$$

For example, this could be a translog production function or a unit cost function. Since economic theory often suggests that such a function should enjoy a concavity property, we may wish to test an hypothesis of the form $H_0 : \Delta \leqslant 0$ (Δ negative semidefinite), where $\Delta = [\delta_{ij}]_{i,j=1,2}$ and $\delta_{21} = \delta_{12}$, or equivalently $H_0 : \delta_{11} \leqslant 0$, $\delta_{22} \leqslant 0$, and $\delta_{11}\delta_{22} - \delta_{12}^2 \geqslant 0$, against the unrestricted model. Here the null hypothesis imposes a nonlinear inequality and two linear inequalities. Since $k = 6$ and the restrictions involve 3 of the parameters ($q = 3$), we can use the conservative critical value $Q_\alpha = 3F_\alpha(3, n-6)/(n-6)$. Suppose now that $\Delta \leqslant 0$ is a maintained restriction under both the null and the alternative hypotheses, and we wish to test the hypotheses $H_0^* : \delta_{12} = 0$ and $H_0^{**} : \gamma_1 + \gamma_2 = 1$. The first problem is equivalent to testing $H_0^* : \delta_{12} = 0$, $\delta_{11} \leqslant 0$, $\delta_{22} \leqslant 0$, and $\delta_{11}\delta_{22} - \delta_{12}^2 \geqslant 0$, against $H_1^* : \delta_{11} \leqslant 0$, $\delta_{22} \leqslant 0$, and $\delta_{11}\delta_{22} - \delta_{12}^2 \geqslant 0$. Thus $q = 3$ and $Q_\alpha = 3F_\alpha(3, n-6)/(n-6)$ is a conservative critical value while, by Corollary 2 (with $p = 1$ and $d = 0$), $Q'_\alpha = F_\alpha(1, n-6)/(n-6)$ is a liberal critical value. The second problem is equivalent to testing $H_0^{**} : \gamma_1 + \gamma_2 = 1$, $\delta_{11} \leqslant 0$, $\delta_{22} \leqslant 0$, and $\delta_{11}\delta_{22} - \delta_{12}^2 \geqslant 0$, against H_1^*. Then $q = 4$ (the restrictions can be formulated in terms of 4 different linear transformations of the model coefficients, i.e., $\gamma_1 + \gamma_2$, δ_{11}, δ_{22}, δ_{12}), $Q_\alpha = 4F_\alpha(4, n-6)/(n-6)$ is a conservative critical value, while $Q'_\alpha = F_\alpha(1, n-6)/(n-6)$ is a liberal one.

As pointed out above, a critical value $\bar{Q}_\alpha$ may exist such that $\bar{Q}_\alpha < Q_\alpha$ and (2.12) holds ($\bar{Q}_\alpha$ is a tight critical value). Similarly, the inequality (2.15) may hold for a critical bound larger than Q'_α. For nonlinear hypotheses, however, a tight critical value (or points closer to it) may be very difficult to compute. In such situations, the only exact critical bounds available are those given by Corollary 1 and, when applicable, by Corollary 2. Indeed, in cases that do not reduce to the usual F test, tight critical values have been developed only for special problems.

An interesting situation where a tight critical value can be found (at least in principle) is the problem of testing hypotheses that involve linear inequality restrictions: $H_0 : C\beta = \gamma_0$ against $H_1 : C\beta \geqslant \gamma_0$, or H_1 against $H_2 : \beta \in \mathbb{R}^k$, where γ_0 is a $q \times 1$ fixed vector. Exact tests for such problems have been studied by Gouriéroux and Monfort (1979), Yancey, Judge, and Bock (1981), Yancey, Bohrer, and Judge (1982), Gouriéroux, Holly, and Monfort (1982), Farebrother (1984, 1986), Hillier (1986), Judge and Yancey (1986), and Wolak (1987).[8] The problem of testing hypotheses of the form $\bar{H}_0 : C_1\beta = \gamma_{10}$ and $C_2\beta = \gamma_{20}$, against

[8] These authors give finite-sample results for the classical linear model (in some cases, σ^2 is taken as known). Further recent results on inference in linear models with linear inequalities are available in King and Smith (1986) and Geweke (1987). For some asymptotic results applicable under more general assumptions, see Gouriéroux, Holly, and Monfort (1980), Kodde and Palm (1986, 1987), Rogers (1986), and Wolak (1985).

$\bar{H}_1 : C_1\beta \geqslant \gamma_{10}$ and $C_2\beta = \gamma_{20}$, or $\bar{H}_1$ against $H_2 : \beta \in \mathbb{R}^k$, is also considered by the same authors. In these problems, the null distribution of the LR statistic is basically a mixture of Fisher (or chi-square) distributions. Except for special cases where closed-form solutions are available (see Kudo (1963), Yancey et al. (1981), Shapiro (1985)), the weights of the mixture must be obtained by evaluating multivariate normal probabilities (which depend on X and C). Algorithms to find these weights are described by Bohrer and Chow (1978) and Farebrother (1984, 1986). It is also possible to obtain approximate weights by using simulation techniques (see Gouriéroux et al. (1982, p. 78)). However, these computations can be costly, especially when the number of linear inequalities is large. Thus, even in the case of linear inequalities, the bounds given by Corollaries 1 and 2 may be useful. They are practically costless. When the weights are relatively difficult to find, we recommend that the bounds be first checked; if the result is inconclusive, a tighter critical value (or a p-value) may be computed by a more costly method. Note also that tighter bounds could be derived from the results of Perlman (1969) and Kodde and Palm (1986). For example, to test r equalities and q inequalities on linear functions of the parameters, the asymptotic null distribution of the statistic $n[\log(SS_0/SS_1)]$ is a mixture of the chi-square distributions $\chi^2(r)$, $\chi^2(r+1), \ldots, \chi^2(r+q)$, which is bounded by the mixtures $\frac{1}{2}\chi^2(r) + \frac{1}{2}\chi^2(r+1)$ and $\frac{1}{2}\chi^2(r+q-1) + \frac{1}{2}\chi^2(r+q)$; see Kodde and Palm (1986). In finite samples, analogous bounds would involve Fisher distributions rather than chi-square distributions.

The problem of comparing two non-nested linear models is also noteworthy. Consider the models

$$(3.1) \qquad H_0 : y = X_0\beta_0 + \bar{u}_0, \quad H_1 : y = X_1\beta_1 + \bar{u}_1,$$

where X_i is a fixed $n \times k_i$ matrix with $1 \leqslant \text{rank}(X_i) = k_i < n$, β_i is a $k_i \times 1$ coefficient vector and $\bar{u}_i \sim N[0, \sigma_i^2 I_n]$ with $\sigma_i^2 > 0$ unknown $(i = 0, 1)$. Suppose also that $X_0 = [\bar{X}_0, Z]$ and $X_1 = [\bar{X}_1, Z]$ where Z is an $n \times m$ matrix of common regressors $(0 \leqslant m \leqslant k_i)$, and $\text{rank}(X) \equiv k = k_0 + k_1 - m < n$, where $X \equiv [\bar{X}_0, \bar{X}_1, Z]$. We wish to test H_0 against H_1.

It is clear that H_0 and H_1 are special cases of the comprehensive model

$$(3.2) \qquad y = X\delta + u = \bar{X}_0\delta_0 + \bar{X}_1\delta_1 + Z\delta_2 + u,$$

where $u \sim N[0, \sigma^2 I_n]$: $\delta_1 = 0$ yields H_0, and $\delta_0 = 0$ yields H_1. From Corollary 1, we have under H_0

$$(3.3) \qquad P[(SS_0 - SS_1)/SS_1 \geqslant (k_1 - m)F_\alpha(k_1 - m, n - k)/(n - k)] \leqslant \alpha,$$

where $SS_i = \text{Min}_{\bar{\beta}_i} \|y - X_i\bar{\beta}_i\|^2$, $i = 0, 1$; or equivalently

$$(3.4) \qquad P[(SS_1/SS_0) \leqslant C_\alpha] \leqslant \alpha,$$

where $C_\alpha = \{1 + [(k_1 - m)F_\alpha(k_1 - m, n - k)/(n - k)]\}^{-1}$. The likelihood ratio test for two non-nested linear models was studied recently by King (1985), who points out that the test cannot be similar and shows how to compute the exact distribution function of the likelihood ratio.[9] In this case, exact critical values which are tighter than those supplied by Corollary 1 may be obtained by using the Imhof (1961) algorithm. Note however that finding these critical values (which depend on each particular regressor matrix) or computing p-values may require a fair degree of computation. Before undertaking this operation, it is certainly worthwhile to check whether the bound given by (3.3) or (3.4) yields a significant result. Note that Corollary 2 is not applicable in this case.

As an illustration, consider critical regions of the form $SS_1/SS_0 < c$, and take $k_1 = k_2 = 2$, $m = 1$, and $\alpha = .05$. Then, for $n = 20$, we find as lower bound on the critical value $C_{.05} = 0.793$; for three different regressor matrices with the same dimensions, King (1985, Table 1) finds 0.8547, 0.8645, and 0.8334 respectively as the critical values that yield tests of size .05. For $n = 60$, we find the lower bound $C_{.05} = 0.934$ while King finds the critical values 0.9537, 0.9573, and 0.9461. In these examples, the lower bound and the tight critical values are remarkably close. Though there is no general guarantee that this will be the case, this proximity underscores the fact that the bounds can be quite useful in practice. When computing tight critical values is considered costly, we thus recommend that the bound on the critical value be first checked; if the test statistic is not found significant, then the critical value (or the p-value) may be computed with the method described by King (1985).

When the alternatives considered involve linear inequalities or non-nested linear hypotheses, algorithms are available to compute tight critical values. In these situations, the bounds given in Corollaries 1 and 2 can still be useful to save computation costs. But clearly these are relatively regular situations. When nonlinear hypotheses of a more complex form are involved, no algorithm seems available. In all cases, the bounds given by Corollary 1 are applicable and, in an important subset of cases, those of Corollary 2 can also be used.

4. MULTIPLE TESTING

The bounds given by Theorems 1 and 2 have another property which is not shared by other exact critical values (if available): they can be used for simultaneous inference. In this section, we study this property.

Here, rather than a single LR statistic associated to (Γ_0, Ω_1), we consider a family of LR statistics associated with a set of comparisons $H(J) = \{(\Gamma_{a0}, \Omega_{a1}) : a \in J\} \subseteq H$. The index set may be fixed or data-dependent (as happens, for example, if pre-testing is taking place), finite or infinite, countable or uncountable. Clearly, in most practical situations, J is finite. From Theorem 1,

[9] The tests studied by Cox (1961, 1962) and the various extensions of these, e.g. Pesaran (1974), Pesaran and Deaton (1978), Davidson and MacKinnon (1981), Fisher and McAleer (1981), Godfrey (1983), or Mizon and Richard (1986), are not *stricto sensu* likelihood ratio tests (between two *non-nested* hypotheses). For reviews of non-nested hypothesis tests, see MacKinnon (1983) and McAleer (1987).

the important result for simultaneous inference is the second inequality in (2.2). From the latter, we see that

(4.1) $P[T_a \geqslant Q_\alpha$, for some $a \in J] \leqslant \alpha$.

This property has two basic interpretations.

The first interpretation allows one to construct finite-sample induced tests of nonlinear hypotheses. Suppose that $\varnothing \neq \Gamma_0 \subseteq \bigcap_{a \in J} \Gamma_{a0}$ (with probability 1) and let $H_0: C\beta \in \Gamma_0$ be a null hypothesis of interest. Then the induced test that rejects H_0 when $T_a > Q_\alpha$ for at least one $a \in J$, has size not greater than α.[10] The null hypotheses tested (or the contrasts considered) are typically *chosen* so that $\Gamma_0 \subseteq \bigcap_{a \in J} \Gamma_{a0}$. This choice may be influenced by the data (for example, through pretesting to select the sets Γ_{a0}).

The second interpretation follows the spirit of multiple comparison methods. Suppose that the collection of null hypotheses $K(J) = \{\Gamma_{a0}: a \in J\}$ on $C\beta$ may contain both true and false hypotheses. Then using the critical value Q_α makes sure that the probability of rejecting a true hypothesis (or more) is not greater than α. Or, equivalently, the probability that *all* the rejected hypotheses be false is not smaller than $1 - \alpha$. The fact that this property is implied by (2.2) can be seen as follows.

Let $J_1 = \{a \in J: C\beta \in \Gamma_{a0}\}$ the indices in J associated to true hypotheses (β is the true coefficient vector) and $\gamma = C\beta$. Setting $\operatorname{Sup}\{T_a: a \in J_1\} = 0$ when $J_1 = \varnothing$, the probability of rejecting a true hypothesis (or more) in $K(J)$ is $P[\operatorname{Sup}\{T_a: a \in J_1\} > Q_\alpha]$. Consider the collection of hypothesis pairs obtained by adding the pair $(\{\gamma\}, \mathbb{R}^k)$ to $H(J_1)$:

$$H(\bar{J}_1) = \{(\Gamma_{a0}, \Omega_{a1}): a \in J_1\} \cup \{(\{\gamma\}, \mathbb{R}^k)\} = \{(\Gamma_{a0}, \Omega_{a1}): a \in \bar{J}_1\},$$

where $\bar{J}_1$ is the appropriately extended index set. Clearly $J_1 \subseteq \bar{J}_1$ and $C\beta \in \bigcap_{a \in \bar{J}_1} \Gamma_{a0}$ with probability 1. Thus

(4.2) $P[\operatorname{Sup}\{T_a: a \in J_1\} > Q_\alpha] \leqslant P[\operatorname{Sup}\{T_a: a \in \bar{J}_1\} > Q_\alpha] \leqslant \alpha$,

where the second inequality follows from Theorem 1.

Let us rewrite the criterion $(SS_{a0} - SS_{a1})/SS_{a1} > Q_\alpha$ as $(SS_{a0} - SS_{a1})/s_{a1}^2 > S_\alpha^2$, where $s_{a1}^2 = SS_{a1}/(n-k)$ and $S_\alpha^2 = qF_\alpha(q, n-k)$. We see immediately that S_α is the critical value used in Scheffé's multiple comparison procedure; see Scheffé (1953; 1959, p. 69) and Savin (1984, p. 849). The relationship with Scheffé's method is indeed much deeper. Consider a linear hypothesis of the form $l_a'\beta = c_\alpha$, where $l_a \in \mathbb{R}^k$ and $c_a \in \mathbb{R}$, and let the alternative hypothesis be $\Omega_{a1} =$

[10] It is important to note here that the test statistics T_a, $a \in J$, are not generally independent. If they are independent and the distribution of each statistic under the null hypothesis is known, it is easy to find a tight critical value for the corresponding induced test. For some interesting recent work which exploits the independence of test statistics (for misspecification testing), see Phillips and McCabe (1984) and Kiviet and Phillips (1985).

$\mathbb{R}^k$. Then

$$(4.3) \qquad (SS_{a_0} - SS_{a1})/s_{a1}^2 = \left[\frac{l_a'\hat{\beta} - c_a}{s\left[l_a'(X'X)^{-1}l_a\right]^{1/2}} \right]^2 = t(l_a, c_a)^2$$

where $s^2 = \|y - X\hat{\beta}\|^2/(n-k)$ and $\hat{\beta} = (X'X)^{-1}X'y$. In this case, the significance criterion for each a can be written $|t(l_a, c_a)| > S_\alpha$. Scheffé (1953, 1959) showed that

$$(4.4) \qquad P\left[|t(l, l'\beta)| \leqslant S_\alpha, \forall\, l \in L_q\right] = 1 - \alpha,$$

where L_q is any linear subspace of $\mathbb{R}^k$ with dimension q; or equivalently

$$(4.5) \qquad P\left[\text{Sup}\left\{|t(l, l'\beta)| : l \in L_q\right\} > S_\alpha\right] = \alpha.$$

An important practical implication of (4.5) is

$$(4.6) \qquad P\left[\text{Sup}\left\{|t(l, l'\beta)| : l \in M\right\} > S_\alpha\right] \leqslant \alpha$$

for any nonempty subset $M \subseteq L_q$. The latter shows that S_α may be used as a simultaneous critical value for any collection of hypotheses of the form $H_{a0}: l_a'\beta - c_a = 0$, where $l_a \in L_q$, against the unrestricted model. On the other hand, Theorem 1 above shows that Scheffé's critical values can be applied to any collection of hypotheses of the form $H_{a0}: C\beta \in \Gamma_{a0}$, where $\Gamma_{a0} \subseteq \mathbb{R}^q$. There is no other restriction on their form. Both the finite-sample and simultaneity properties are preserved.

When we test a *finite* number of fixed linear hypotheses (say, m), it is also possible to find a simultaneous critical value by using Bonferroni inequality. One simply needs to use α/m as the level for each of the m separate t tests: the induced test that rejects the joint hypothesis when at least one t statistic is significant at level α/m has size no greater than α. When m is small, the critical value obtained in this way (call it $t_{\alpha/m}$) can be smaller than S_α. However, it is easy to see that $t_{\alpha/m}$ increases with m (while S_α does not depend on m), so that $t_{\alpha/m}$ must be larger than S_α for m large. For further discussion, see Miller (1966) and Savin (1984). Note also that, for nonlinear hypotheses, exact Bonferroni bounds are not available because the marginal distributions of individual LR statistics are unknown.

The first inequality in (2.2) implies that the critical region $T_b > Q_\alpha$ has size not greater than α for testing $C\beta \in \Gamma_0$, where $\Gamma_0 \subseteq \bigcap_{a \in J} \Gamma_{a0}$, irrespective of the way b is selected in J. For example, if the test statistics associated with $H(J)$ are computed one after the other, b may be the first statistic greater than Q_α. This observation can be useful from a computational point of view because, in this case, one does not need to find the supremum of the test statistics.

Similarly, since minimizing the sum of squares under nonlinear restrictions can be costly, the simultaneity property of Q_α remains unaffected if SS_{a1} is replaced by $S(\tilde{\beta}_a)$, where $\tilde{\beta}_a$ is any estimator of β that satisfies the alternative hypothesis Ω_{a1}. A precise statement of this observation is given by the following corollary of

Theorem 1, which can be viewed as a generalization of Corollary 1.1 and follows in the same way.

COROLLARY 3: *Let the assumptions of Theorem* 1 *hold and let* $\{\tilde{\beta}_a : a \in H\}$ *be a family of* $k \times 1$ *random vectors such that* $P[\tilde{\beta}_a \in \Omega_{a1}] = 1$ *for all* $a \in H$ *and for all* β. *Then, if* $C\beta \in \bigcap_{a \in J} \Gamma_{a0}$ (*with probability* 1), *we have*

$$(4.7) \qquad P\big[\{SS_{b0} - S(\tilde{\beta}_b)\}/S(\tilde{\beta}_b) \geqslant Q_\alpha\big]$$
$$\leqslant P\big[\mathrm{Sup}\,\{[SS_{a0} - S(\tilde{\beta}_a)]/S(\tilde{\beta}_a) : a \in J\} \geqslant Q_\alpha\big]$$
$$\leqslant P\big[\mathrm{Sup}\,\{T_a : a \in J\} \geqslant Q_\alpha\big] \leqslant \alpha,$$

where the index b can be chosen in J by any rule.

Theorem 2 gives a general sufficient condition under which a *liberal critical region* may be obtained from a *collection of test statistics* (instead of only one statistic). To understand its meaning, suppose that we want to test $H_0 : \beta \in \Omega_0$, where $\Omega_0 \subseteq \bigcap_{a \in J} \Omega_{a0}$ (with probability 1), and consider the test that rejects H_0 when $\mathrm{Inf}\,\{T_a : a \in J\}$ is large. If the conditions of Theorem 2 hold, we know from (2.13) that the critical region $\mathrm{Inf}\{T_a : a \in J\} > Q'_\alpha$ has size not smaller than α. Thus the result $\mathrm{Inf}\{T_a : a \in J\} \leqslant Q'_\alpha$ is certainly not significant at level α. When J is finite, we can say also that the test which accepts H_0 when $T_a \leqslant Q'_\alpha$ for some $a \in J$ has size not smaller than α. A similar interpretation holds for the modified statistics considered in the following corollary of Theorem 2, which can be viewed as a generalization of Corollary 2.1.

COROLLARY 4: *Let the assumptions of Theorem* 2 *hold and let* $\{\tilde{\beta}_a \; a \in \tilde{H}\}$ *be a family of* $k \times 1$ *random vectors such that* $P[\tilde{\beta}_a \in \Omega_{a0}] = 1$ *for all* $a \in \tilde{H}$ *and for all* β. *Then, if* $\beta \in \bigcap_{a \in J} \Omega_{a0}$ (*with probability* 1), *we have*

$$(4.8) \qquad P\big[\{S(\tilde{\beta}_b) - SS_{b1}\}/SS_{b1} > Q'_\alpha\big]$$
$$\geqslant P\big[\mathrm{Inf}\{[S(\tilde{\beta}_a) - SS_{a1}]/SS_{a1} : a \in J\} > Q'_a\big]$$
$$\geqslant P\big[\mathrm{Inf}\{T_a : a \in J\} > Q'_\alpha\big] \geqslant \alpha,$$

where the index b may be chosen in J by any rule.

When Theorem 2 is applicable, it is natural to combine it with Theorem 1 and seek a *bounds induced test* for $H_0 : \beta \in \Omega_0$. However we must observe that there is usually a nonzero probability that the events $\mathrm{Sup}\,\{T_a : a \in J\} > Q_\alpha$ and $\mathrm{Inf}\,\{T_a : a \in J\} \leqslant Q'_\alpha$ occur together. In other words, the two procedures could give conflicting answers. There is a simple way to avoid this difficulty. Consider the test that rejects H_0 when $\mathrm{Inf}\,\{T_a : a \in J\} > Q'_\alpha$ or $\mathrm{Sup}\,\{T_a : a \in J\} > Q_\alpha$. Clearly

$$(4.9) \qquad P\big[\mathrm{Inf}\,\{T_a : a \in J\} > Q'_\alpha \text{ or } \mathrm{Sup}\,\{T_a : a \in J\} > Q_\alpha\big]$$
$$\geqslant P\big[\mathrm{Inf}\,\{T_a : a \in J\} > Q'_\alpha\big] \geqslant \alpha$$

when $\beta \in \Omega_0$. Thus, the test that accepts H_0 when $\mathrm{Inf}\,\{T_a : a \in J\} \leqslant Q'_\alpha$ and $\mathrm{Sup}\,\{T_a : a \in J\} \leqslant Q_\alpha$ has size not smaller than α.

If $\Omega_0 \subseteq \bigcap_{a \in J} \Omega_{a0}$ (with probability 1), a bounds induced test of $H_0: \beta \in \Omega_0$ may be built as follows:

(4.10) reject H_0 if $\operatorname{Sup}\{T_a : a \in J\} > Q_\alpha$,

accept H_0 if $\operatorname{Sup}\{T_a : a \in J\} \leqslant Q_\alpha$ and $\operatorname{Inf}\{T_a : a \in J\} \leqslant Q'_\alpha$,

test inconclusive, otherwise.

When Theorem 2 applies, this procedure satisfies the same basic conditions as the procedure in (2.19), namely

$$P[\text{Rejecting } H_0 | H_0] \leqslant \alpha, \quad P[\text{Accepting } H_0 | H_0] \leqslant 1 - \alpha.$$

It can be viewed as a *generalized bounds test* (see Dufour (1986)). Clearly (2.19) is a special case of (4.10) where J contains only one pair of hypotheses. When J is finite, the procedure can also be described as follows:

(4.11) reject H_0 if $T_a > Q_\alpha$ for some $a \in J$,

accept H_0 if $T_a \leqslant Q_\alpha$ for all $a \in J$ and $T_b \leqslant Q'_a$ for some $b \in J$,

test inconclusive, otherwise.

When applicable, the bounds procedure is a natural modification of the conservative induced test. Even in the case of linear hypotheses, this possibility was not apparently pointed out in the previous literature on simultaneous inference (see Miller (1966, 1977) and Savin (1980, 1984)).

5. CONCLUSION

In this paper, we have shown that a critical value based on the central Fisher distribution may always be used as a conservative critical bound for any likelihood ratio test on the coefficients of a standard linear regression. The bounds so obtained are exact, easy to compute, applicable to any pair of hypotheses and enjoy a strong simultaneity property. Even when a problem is highly nonregular, a bound is always available. Many difficult cases frequently met in econometric work are covered: nonlinear hypotheses, inequality restrictions (linear or nonlinear), non-nested hypotheses, etc. Of course, one should seek the smallest critical bound possible. In particular, it is advantageous to express the null hypothesis in terms of a small number of linear coefficients, i.e. in terms of a $q \times 1$ vector $\gamma = C\beta$ where q is as small as possible. In all cases, however, the bound with $q = k$ is valid. In certain situations (for example, nested hypotheses where the null imposes some linear restrictions), it is also possible to find a lower bound on the tight critical value. Then both an upper and a lower bound on the null distribution of the likelihood ratio are available, and a bounds test similar to the Durbin-Watson test may be used. Since they are very easy to compute, the bounds may be useful even in those nonregular situations where algorithms are available to compute tight critical values (linear inequalities, non-nested linear models).

The simultaneity property is analogous to the simultaneity of Scheffé's (1953, 1959) critical values for t statistics. Indeed our critical bounds are basically the same as Scheffé's: through a different (and simpler) method of proof, we have found that the result given by Scheffé for collections of simple linear hypotheses about a vector $\gamma = C\beta$ holds for any collection of hypotheses about γ. Thus we can control the level α of the induced test associated with any set of comparisons between a null hypothesis about γ and some alternative on γ (or β). In this way, for any family of hypotheses about γ (some of which are true, some of which are false), the probability that all the rejected hypotheses are false is not smaller than $1-\alpha$. Finally, we have given conditions under which a bounds induced test can be obtained.

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