Reliable inference for inequality measures with heavy-tailed distribution

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Motivation

In the presence of heavy-tails:

- Asymptotic and bootstrap inference perform poorly in finite sample
- Alternative methods: improvements, but inference is still unreliable

<table>
<thead>
<tr>
<th></th>
<th>asym</th>
<th>boot</th>
<th>varstab$^1$</th>
<th>semip$^2$</th>
<th>mixture$^3$</th>
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</thead>
<tbody>
<tr>
<td>Singh-Maddala</td>
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<tr>
<td>$q = 1.7$</td>
<td>0.915</td>
<td>0.931</td>
<td>0.933</td>
<td>0.926</td>
<td>0.928</td>
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<tr>
<td>$q = 1.2$</td>
<td>0.856</td>
<td>0.905</td>
<td>0.899</td>
<td>0.905</td>
<td>0.912</td>
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<td>$q = 0.7$</td>
<td>0.647</td>
<td>0.802</td>
<td>0.796</td>
<td>0.871</td>
<td>0.789</td>
</tr>
</tbody>
</table>

Table: Coverage of asymptotic and bootstrap confidence intervals at the 95% level for the Theil index, for several bootstrap approaches, $n = 500$.

Note that it is a large sample problem.

$^1$Schluter and van Garderen (2009, JoE)
$^2$Davidson and Flachaire (2007, JoE), Cowell and Flachaire (2007 JoE)
$^3$Cowell and Flachaire (2013, Handbook)
Our approach

We are interested in testing

\[ H_0 : \theta(F_x) = \theta(F_y) \]  

(1)

Monte Carlo permutation tests:

- If \( F_x = F_y \), a permutation test provides exact inference. ⁴
- For a nominal level \( \alpha \), critical value or \( p \)-value obtained from the permutation distribution would then be similar than those obtained from the true distribution.
- There is no need to obtain the permutation distribution from all the possible permutations

Problem: (1) does not guarantee that \( F_x = F_y \). Two different distributions can share the same inequality index.

Our approach

The following Figure depicts Singh-Maddala distributions (Burr XII).\(^5\) One distribution is much more heavy-tailed than the other, yet both distributions share the same value of the Theil index.

\[^5\text{with density } f(u) = aqu^{a-1}/(b^a[1 + (u/b)^a]^{1+q}), \text{ for two choices of } a, b \text{ and } q: 2.8, 0.1930698, 1.7 \text{ [depicted as } F_x\] and 4.8, 0.1930698, 0.6366578 \text{ [depicted as } F_y].\]
Our approach

We are interested in testing

$$H_0 : \theta(F_x) = \theta(F_y) \quad (1)$$

The use of permutation tests is not justified from an exact perspective. We thus analyze the asymptotic validity of permutation tests of (1) when $F_x \neq F_y$.

- We show that permutation tests can be used reliably with the most popular inequality measures provided considered samples are recentered or rescaled adequately.
- A bootstrap method that respects the null hypothesis is also proposed.
- Simulation experiments are provided to study the finite sample properties
Outline

1 Finite and large sample theory
   - Exact inference
   - Asymptotic validity
   - Bootstrapping under the null

2 Comparing inequality measures
   - Centered and uncentered moments
   - The generalised entropy class (Theil index)
   - The Gini coefficient

3 Simulation study
   - Model design
   - Results
Permutation test

- $X = \{X_1, X_2, \ldots, X_n\} \sim F_x$ and $Y = \{Y_1, Y_2, \ldots, Y_m\} \sim F_y$.
  We test the null $H_0 : \theta(F_x) = \theta(F_y)$, with the statistic
  \[
  T(X, Y) = \sqrt{n} \left[ \theta(\hat{F}_x) - \theta(\hat{F}_y) \right].
  \]

- The permutation distribution is obtained by permuting in all possible ways the $n + m$ observations of the combined sample
  \[
  Z = \{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m\}.
  \]
  It is the distribution of the permutation statistic, defined as
  \[
  T_\ast = \sqrt{n} \left[ \theta(\hat{F}_{x\ast}) - \theta(\hat{F}_{y\ast}) \right],
  \]

\[\text{\footnotesize{6}}\hat{F}_{x\ast}\text{ and }\hat{F}_{y\ast}\text{ are the EDF of, respectively, the first } n\text{ and the remaining } m\text{ observations of a permuted sample.}\]
Asymptotic validity

Romano (1990) shows that the permutation test is asymptotically valid if the asymptotic variances of the original and permutation statistics are similar, that is,

\[ V[\theta(\hat{F}_w)] = (1 - \lambda) V[\theta(\hat{F}_x)] + \lambda V[\theta(\hat{F}_y)] \]

where \( \hat{F}_w = \lambda \hat{F}_x + (1 - \lambda) \hat{F}_y \). Let \( w \sim \sum_{k=1}^{K} \lambda_k F_k(w) \) and let \( w_1, \ldots, w_K \) denote random variables from the \( K \) component dist.

\[ V[\theta(\hat{F}_w)] = E \left[ \left( \theta(\hat{F}_w) - E[\theta(\hat{F}_w)] \right)^2 \right] \]

\[ = \sum_{k=1}^{K} \lambda_k E \left[ \left( \theta(\hat{F}_{w_k}) - E[\theta(\hat{F}_{w_k})] + E[\theta(\hat{F}_{w_k})] - E[\theta(F_w)] \right)^2 \right] . \]

\[ = \sum_{k=1}^{K} \lambda_k V[\theta(\hat{F}_{w_k})] \quad \text{if} \quad E[\theta(\hat{F}_{w_k})] = E[\theta(\hat{F}_w)], \forall k. \]
Asymptotic validity

Result

A permutation test is asymptotically valid if, under the null hypothesis, the two distributions \( F_x, F_y \) and the mixture distribution \( F_w \) share the same value of the statistic

\[
\theta(F_w) = \theta(F_x) = \theta(F_y)
\]

where

\[
F_w = \lambda F_x + (1 - \lambda) F_y
\]

and, either \( n/(n + m) \to \lambda = 1/2 \) or \( V[\theta(\hat{F}_x)] = V[\theta(\hat{F}_y)] \)

Permutation test is as. valid if the index is the same in \( F_x, F_y, F_w \)
The generalised entropy class of inequality measures

\[ I_{GE}^{\zeta}(F) = \frac{1}{\zeta^2 - \zeta} \left[ \int \left( \frac{y}{\mu(F)} \right)^\zeta dF(y) - 1 \right] , \ \zeta \in \mathbb{R}, \zeta \neq 0, 1 \]

\[ I_{GE}^0(F) = -\int \log \left( \frac{y}{\mu(F)} \right) dF(y) \]

\[ I_{GE}^1(F) = \int \frac{y}{\mu(F)} \log \left( \frac{y}{\mu(F)} \right) dF(y) \]

- \( I_{GE}^0(F) \) is the Mean Logarithmic Deviation index (\( \zeta = 0 \))
- \( I_{GE}^1(F) \) is the Theil index (\( \zeta = 1 \)).
- The more positive \( \zeta \) is, the more sensitive is the inequality measure to income differences at the top of the distribution.
A decomposable class of measures

- The GE inequality measure is decomposable by groups:

\[ I_{GE}^{\xi}(\hat{F}_w) = \sum_{k=1}^{K} I_{GE}^{\xi}(\hat{F}_{w_k}) + I_{between}^{\xi} \]

where \( I_{between}^{\xi} = 0 \) when the groups share a common mean.

- Then, permutation test is asymptotically valid if

\[ \mu(F_x) = \mu(F_y) \]

This condition does not hold in general.
Rescaled samples

- The GE inequality measures are scale invariant: calculating indices from the original samples or from the rescaled samples
  \[
  \left\{ \frac{X_1}{\mu(F_X)}, \ldots, \frac{X_n}{\mu(F_X)} \right\} \quad \text{and} \quad \left\{ \frac{Y_1}{\mu(F_Y)}, \ldots, \frac{Y_m}{\mu(F_Y)} \right\},
  \]
  gives similar results

- The rescaled samples have a common mean, equals to one.

- Permutation test is asymptotically valid, when based on
  \[
  \left\{ \frac{X_1}{\mu(F_X)}, \ldots, \frac{X_n}{\mu(F_X)}, \frac{Y_1}{\mu(F_Y)}, \ldots, \frac{Y_m}{\mu(F_Y)} \right\}.
  \]

- In practice, population means are replaced by sample means
Bootstrapping under the null

\[ Z_s = \left\{ \frac{X_1}{\bar{X}}, \ldots, \frac{X_n}{\bar{X}}, \frac{Y_1}{\bar{Y}}, \ldots, \frac{Y_m}{\bar{Y}} \right\} \]

Permutation approach:

- resample \textit{without} replacement \( n \) observations in \( Z_s \) to form \( X_* \)
- the \( m \) remaining observations in \( Z_s \) are then used to form \( Y_* \)
- compute the statistic from \( X_* \) and \( Y_* \)

Bootstrap approach:

- resample \textit{with} replacement \( n \) observations in \( Z_s \) to form \( X_b \)
- resample \textit{with} replacement \( m \) observations in \( Z_s \) to form \( Y_b \)
- compute the statistic from \( X_b \) and \( Y_b \)

The bootstrap respects the null (resample from same set of obs.)
We test the null $H_0 : \theta(F_x) = \theta(F_y)$ with a two-tailed $t$-statistic

$$T = \frac{\theta(\hat{F}_x) - \theta(\hat{F}_y)}{\sqrt{V[\theta(\hat{F}_x) - \theta(\hat{F}_y)]}}$$

We consider the following methods:

- asymptotic test
- (standard) bootstrap test
- permutation test based on the combined sample $Z^s$
- bootstrap under the null based on the combined sample $Z^s$

We compare Theil indices based on Singh-Maddala distributions
Simulation results in very small sample

Figure: Rejection frequencies for the Theil index as the upper tail is heavier (as $\xi_y$ decreases). Left panel: $F_x = F_y$. Right panel: $F_x \neq F_y$. $n = 20$, $B = 999$, $R = 10000$, $\alpha = 0.05$. 

Simulation results as the sample size increases

Figure: Rejection frequencies for the Theil inequality index, in the worst case, as the sample size increases. Left panel: $F_x = F_y$. Right panel: $F_x \neq F_y$. $B = 999$, $R = 10000$, $\alpha = 0.05$. 
Simulation results show that when the samples are drawn from two (strongly) heavy-tailed distributions which are not too different, the permutation approach and the proposed bootstrap that respects the null hypothesis perform very well in finite samples.

When distributions differ dramatically particularly in their tails, while size distortions are not completely eradicated, our proposed methods outperform the standard asymptotic and bootstrap tests.