Discussion: Reliable inference for inequality measures with heavy-tailed distributions by Jean-Marie Dufour, Emmanuel Flachaire, Lynda Khalaf

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Research question and approach

Issue:

- Let $(X_1, \ldots, X_n)$, $X \sim F_x$, $(Y_1, \ldots, Y_m)$, $Y \sim F_y$, X_is iid, Y_is iid, X and Y independent;
- and let $\theta(.)$ be an inequality index,
- Consider testing $H_0 : \theta(F_x) = \theta(F_y)$.
- Both asymptotic and (classic) bootstrap inference on inequality measures perform poorly in presence of heavy-tailed distributions: income...
- Search for nonparametric methods to improve inference quality

Research question and approach

Idea/Approach:

- Permutation/randomization tests provide exact inference (in finite samples) if under the null, the two samples follow the same distribution, (and more generally when the data are generated from a distribution, which is invariant under some group of transformations.)
- Exploit results of Romano (1990) who derives conditions under which permutation tests are asymptotically valid when under the null $F_x \neq F_y$
- Expect that robustness properties of permutation tests can improve inference quality in more general setups ($F_x \neq F_y$).

Main results

Results

- Show that the equality of Generalized Entropy Indices and Gini Indices can be tested by permutation tests, if the data are well re-scaled
- Propose to use bootstrap under the null
- Provide a simulation comparison study of permutation tests, tests based on bootstrapping under the null, and usual asymptotic and bootstrap tests

Theory

Let $(X_1, \ldots, X_n)$, $X \sim F_x$, $(Y_1, \ldots, Y_m)$, $Y \sim F_y$, X_is iid, Y_is iid, X and Y independent. Consider testing $H_0 : \theta(F_x) = \theta(F_y)$.
With test statistic: $T(X_n, Y_m) = n^{1/2}(\theta(\hat{F_x^n}) - \theta(\hat{F_y^m}))$

Let $Z = (X_1, \ldots, X_n, Y_1, \ldots, Y_m) = (Z_1, \ldots, Z_{n+m})$,
$Z^p = (Z_{p(1)}, \ldots, Z_{p(n)}, Z_{p(n+1)}, \ldots, Z_{p(n+m)})$, where p is a permutation of $(1, n+m)$,
Permutation statistic: $T^p(Z_{n+m}) = n^{1/2}(\theta(\hat{F_1^n}) - \theta(\hat{F_2^m}))$

Romano: $T(X_n, Y_m)$ and $T^p(Z_{n+m})$ follow asymptotically the same distribution under H_0 if their asymptotic variances are the same:

$$Vas(\theta(F_x)) + \frac{\lambda}{1-\lambda} Vas(\theta(F_y)) = \frac{1}{1-\lambda} Vas(\theta(\lambda F_x + (1-\lambda)F_z)) \tag{1}$$

where $\lambda = \frac{n}{n+m}$

Theory

Dufour-Flachaire-Khalaf: A permutation test is asymptotically valid if

- *under* H_0, $\theta(F_x) = \theta(F_y) = \theta(\lambda F_x + (1-\lambda)F_y)$
- $\theta(.)$ *is linear in* F
- *and either* $n/(n+m) \to \lambda = 1/2$, *either* $Vas(\theta(F_x)) = Vas(\theta(F_y))$

Discussion

- Linearity is essential. It allows to express vas of θ of the mixture distribution as a linear combination of vas of θ applied at each component.
- The method is adapted to inequality indexes derived from moments (centered or not, if the data are properly rescaled), such as GEIs, etc.. (but not a quantile ratio, and Gini?)

Simulation study

Setup

$n = m$,$\lambda = .5$, t-stat versions of T_n, $T^* = \frac{\theta(F_x^*) - \theta(F_y^*)}{\sqrt{V}(\theta(F_x^*)) - V(\theta(F_y^*))}$.

Compare

- **boot**: F_x^*, F_y^* obtained drawing n X^* in F_x^n and m Y^* in F_y^m
- **permut**: F_x^*, F_y^* obtained by drawing without replacement, $n + m$ Z^* in F_Z^{n+m}
- **bootH0**: F_x^*, F_y^* obtained by drawing with replacement, $n + m$ Z^* in F_Z^{n+m}

Simulation study

Results: both for Gini and Theil indexes

- When $F_x = F_y$: test sizes are better controlled by **permut** and **bootH0** than **boot** (very heavy upper tail) or **asymptotic** (very heavy upper tail and small n)
- When $F_x \neq F_y$ under H_0: sizes of **permut** and **bootH0** increase when the tails of F_x and F_y differ and n is small, but less than those of **boot** or **asymptotic**.

Questions:

- Gini with $F_x = F_y$: figure 4, **boot** test size is close to .05 when n small and remains constant after. Why ?
- What happens when $\lambda \neq 0.5$? (and $Vas(\theta(F_x)) = Vas(\theta(F_y))$)

Figure : Figure 4

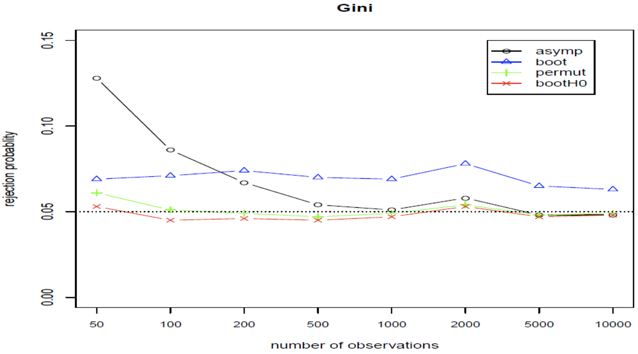

Figure 4: Rejection frequencies for the Gini index, with $F_x = F_y$ in the worst case ($\xi_x = \xi_y = 2.59$), as the sample size increases.

Figure : Figure 5

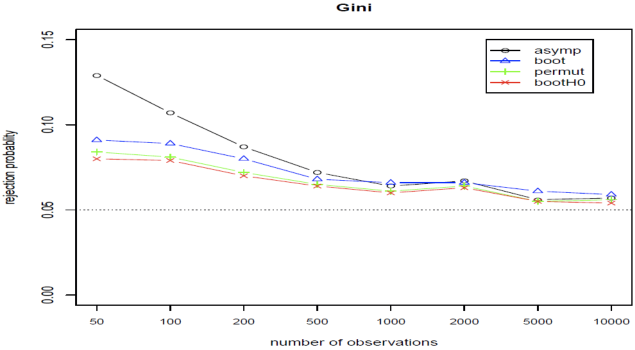

Figure 5: Rejection frequencies for the Gini index, with $F_x \neq F_y$ in the worst case ($\xi_y = 2.59$ and $\xi_x = 4.76$), as the sample size increases.

Questions/Extensions

- What about power comparison?
- May different resampling schemes allow to deal with $\lambda \neq .5$ (and $Vas(\theta(F_x)) \neq Vas(\theta(F_y))$)?
- Extend results when X and Y are correlated ?