Distribution and quantile functions[*]

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1. Monotonic functions

1.1 Definition MONOTONIC FUNCTION. *Let D a non-empty subset of $\mathbb{R}$, $f : D \to E$, where E is a non-empty subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let I be a non-empty subset of D.*

(a) *f is nondecreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \quad \forall x_1, x_2 \in I .$$

(b) *f is nonincreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \quad \forall x_1, x_2 \in I .$$

(c) *f is strictly increasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in I .$$

(d) *f is strictly decreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall x_1, x_2 \in I .$$

(e) *f is monotonic on I iff f is nondecreasing, nonincreasing, increasing or decreasing.*

(f) *f is strictly monotonic on I iff f is strictly increasing or decreasing.*

1.2 Definition MONOTONICITY AT A POINT. *Let D a non-empty subset of $\mathbb{R}$, $f : D \to E$, where E is a non-empty subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let $x \in D$.*

(a) *f is nondecreasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) \le f(x), \quad \forall x_1 \in I \cap D ,$$

$$\text{and } \ x < x_2 \Rightarrow f(x) \le f(x_2), \quad \forall x_2 \in I \cap D ;$$

(b) *f is nonincreasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) \ge f(x), \quad \forall x_1 \in I \cap D ,$$

$$\text{and } \ x < x_2 \Rightarrow f(x) \ge f(x_2), \quad \forall x_2 \in I \cap D ;$$

(c) *f is strictly increasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) < f(x), \quad \forall x_1 \in I \cap D ,$$

$$\text{and } \ x < x_2 \Rightarrow f(x) < f(x_2), \quad \forall x_2 \in I \cap D ;$$

(d) f *is strictly decreasing on* I *iff there is an open neighborhood* I *of* x *such that*

$$x_1 < x \Rightarrow f(x_1) > f(x), \quad \forall x_1 \in I \cap D,$$

$$\text{and } x < x_2 \Rightarrow f(x) > f(x_2), \quad \forall x_2 \in I \cap D.$$

(e) f *is monotonic at* x *iff* f *is nondecreasing, nonincreasing, increasing or decreasing at* x.

(f) f *is strictly monotonic at* x *iff* f *is strictly increasing or decreasing at* x.

1.3 Remark It is clear that:

(a) an increasing function is also nondecreasing;

(b) a decreasing function is also nonincreasing;

(c) if f is nondecreasing (alt., strictly increasing), the function

$$g(x) = -f(x)$$

is nonincreasing (alt., strictly decreasing) on I, and the function

$$h(x) = -f(-x)$$

is nondecreasing on $I_1 = \{x : -x \in I\}$..

1.4 Proposition LIMITS OF MONOTONIC FUNCTIONS. *Let* $I = (a, b) \subseteq \mathbb{R}$, *where* $-\infty \le a < b \le \infty$, *and* $f : I \to \mathbb{R}$ *be a nondecreasing function on* I. *Then the function* f *has the following properties.*

(a) *For each* $x \in (a, b)$, *set*

$$\begin{aligned} f(x_+) &= \lim_{\delta \downarrow 0} \left\{ \inf_{x<y<x+\delta} f(y) \right\}, \ f\left(x^+\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{x<y<x+\delta} f(y) \right\}, \\ f(x_-) &= \lim_{\delta \downarrow 0} \left\{ \inf_{x-\delta<y<x} f(y) \right\}, \ f\left(x^-\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{x-\delta<y<x} f(y) \right\}. \end{aligned}$$

Then, the four limits $f(x_+)$, $f(x^+)$, $f(x_-)$ *and* $f(x^-)$ *are finite and, for any* $\delta > 0$ *such that* $[x-\delta,\, x+\delta] \subseteq (a, b)$,

$$f(x-\delta) \le f(x_-) \le f\left(x^-\right) \le f(x) \le f(x_+) \le f\left(x^+\right) \le f(x+\delta).$$

(b) *For each* $x \in (a, b)$, *we have*

$$f(x_+) = f\left(x^+\right), \ f(x_-) = f\left(x^-\right),$$

and the function $f(x)$ has finite unilateral limits:

$$f(x+) \equiv \lim_{y \downarrow x} f(y) = f(x_+) = f\left(x^+\right) \ , \ f(x-) \equiv \lim_{y \uparrow x} f(y) = f(x_-) = f\left(x^-\right) \ .$$

(c) *For each $x \in (a, b)$,*

$$\sup_{a<y<x} f(y) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<y<b} f(y) \ .$$

(d) *If $a < x < y < b$, then*

$$f(x+) \leq f(y-) \ .$$

(e) *If $a = -\infty$, the function $f(x)$ has a limit in the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ as $x \to -\infty$,*

$$-\infty \leq f(-\infty) \equiv \lim_{x \to -\infty} f(x) < \infty$$

and, if $b = \infty$, the function $f(x)$ has a limit in $\overline{\mathbb{R}}$ as $x \to \infty$:

$$-\infty < f(+\infty) \equiv \lim_{x \to \infty} f(x) \leq \infty.$$

1.5 Theorem CONTINUITY OF MONOTONIC FUNCTIONS. *Let $I = (a, b) \subseteq \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, and $f : I \to \mathbb{R}$ be a nondecreasing function on I. Then the function f has the following properties.*

(a) *For each $x \in (a, b)$, f is continuous at x iff*

$$f(x-) = f(x+) \ .$$

(b) *The only possible kind of discontinuity of f on (a, b) is a jump.*

(c) *The set of points of (a, b) at which f is discontinuous is countable (possibly empty).*

(d) *The function*

$$f_R(x) = f(x+) \ , \quad x \in (a, b)$$

is right continuous at every point of (a, b), i.e.,

$$\lim_{y \downarrow x} f_R(y) = f_R(x) \ , \quad \forall x \in (a, b) \ .$$

(e) *The function*

$$f_L(x) = f(x-)$$

is left continuous at every point of (a, b), i.e.,

$$\lim_{y \uparrow x} f_L(y) = f_L(x) \ , \quad \forall x \in (a, b) \ .$$

1.6 Theorem CHARACTERIZATION OF THE CONTINUITY OF MONOTONIC FUNCTIONS. *Let $f : D \to \mathbb{R}$ a monotonic function, where D is a non-empty subset of $\mathbb{R}$ and I a non-empty subset of D. Then*

$$f \text{ is continuous on } I \text{ iff } f(I) \text{ is an interval.}$$

1.7 Theorem MONOTONE INVERSE FUNCTION THEOREM. *Let I be an interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$. If f is continuous and strictly monotonic, then $J = f(I)$ is an interval and the function $f : I \to J$ is an homeomorphism (i.e., $f : I \to J$ is a bijection such that f and f^{-1} are continuous).*

1.8 Theorem STRICT MONOTONICITY AND HOMEOMORPHISMS BETWEEN INTERVALS. *Let I and J be intervals in $\mathbb{R}$ and $f : I \to J$.*

(a) *If f is an homeomorphism, then f is strictly monotonic.*

(b) *f is an homeomorphism $\Leftrightarrow f$ is continuous and strictly monotonic*
$\Leftrightarrow f^{-1} : J \to I$ exists and is an homeomorphism
$\Leftrightarrow f^{-1} : J \to I$ exists, is continuous and strictly monotonic.

1.9 Lemma CHARACTERIZATION OF RIGHT (LEFT) CONTINUOUS FUNCTIONS BY DENSE SETS. *Let f_1 and f_2 be two real-valued functions defined on the interval (a, b) such that the functions f_1 and f_2 are either both right continuous or both left continuous at each point $x \in (a, b)$, and let D be a dense subset of (a, b). If*

$$f_1(x) = f_2(x), \quad \forall x \in D,$$

then

$$f_1(x) = f_2(x), \quad \forall x \in (a, b).$$

1.10 Theorem CHARACTERIZATION OF MONOTONIC FUNCTIONS BY DENSE SETS. *Let f_1 and f_2 be two monotonic nondecreasing functions on (a, b), let D be a dense subset of (a, b), and suppose*

$$f_1(x) = f_2(x), \quad \forall x \in D.$$

(a) *Then f_1 and f_2 have the same points of discontinuity, they coincide everywhere in (a, b), except possibly at points of discontinuity, and*

$$f_1(x+) - f_1(x-) = f_2(x+) - f_2(x-), \quad \forall x \in (a, b).$$

(b) *If furthermore f_1 and f_2 are both left continuous (or right continuous) at every point $x \in (a, b)$, they coincide everywhere on (a, b), i.e.,*

$$f_1(x) = f_2(x), \quad \forall x \in (a, b).$$

1.11 Theorem DIFFERENTIABILITY OF MONOTONIC FUNCTIONS. *Let $I = (a, b) \subseteq \mathbb{R}$, where $-\infty \le a < b \le \infty$, and $f : I \to \mathbb{R}$ be a nondecreasing function on I. Then f is differentiable almost everywhere on I.*

2. Generalized inverse of a monotonic function

2.1 Definition GENERALIZED INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then the generalized inverse of f is defined by*

$$f^*(y) = \inf\{x \in (a,b) : f(x) \geq y\} \tag{2.1}$$

for $-\infty < y < \infty$ (with the convention $\inf(\emptyset) = b$). Further, we define f^{-1} as the restriction of f^* to the interval $(\inf(f), \sup(f)) \equiv (\inf\{f(x) : x \in (a,b)\}, \sup\{f(x) : x \in (a,b)\})$:

$$f^{-1}(y) = f^*(y) \quad \text{for } \inf(f) < y < \sup(f)\ . \tag{2.2}$$

2.2 Definition GENERALIZED INVERSE OF A NONDECREASING LEFT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then the generalized inverse of f is defined by*

$$f^{**}(y) = \sup\{x \in (a,b) : f(x) \leq y\} \tag{2.3}$$

for $-\infty < y < \infty$ (with the convention $\sup(\emptyset) = a$).

2.3 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (RIGHT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then, for $x \in (a,b)$ and for every real y,*

$$y \leq f(x) \Leftrightarrow f^*(y) \leq x\,, \tag{2.4}$$
$$y > f(x) \Leftrightarrow f^*(y) > x\,, \tag{2.5}$$
$$f[f^*(y)] \geq y\,. \tag{2.6}$$

2.4 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (LEFT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then, for $x \in (a,b)$ and for every real y,*

$$y \leq f(x) \Leftrightarrow f^{**}(y) \geq x\,. \tag{2.7}$$

2.5 Proposition CONTINUITY OF THE INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$, and set*

$$a(f) = \inf\{x \in (a,b) : f(x) > \inf(f)\}\ , \quad b(f) = \sup\{x \in (a,b) : f(x) < \sup(f)\}\ . \tag{2.8}$$

Then, f^* is nondecreasing and left continuous. Moreover

$$\lim_{y \to -\infty} f^*(y) = a\ , \quad \lim_{y \to \infty} f^*(y) = b \tag{2.9}$$

and

$$\lim_{y\to\inf(f)} f^{-1}(y) = a(f)\,, \qquad \lim_{y\to\sup(f)} f^{-1}(y) = b(f)\,. \tag{2.10}$$

3. Distribution functions

3.1 Definition DISTRIBUTION AND SURVIVAL FUNCTIONS OF A RANDOM VARIABLE. *Let X be a real-valued random variable. The distribution function of X is the function $F(x)$ defined by*

$$F(x) = \mathsf{P}[X \le x]\,,\; x \in \mathbb{R}\,, \tag{3.1}$$

and its survival function is the function $G(x)$ defined by

$$G(x) = \mathsf{P}[X \ge x]\,,\; x \in \mathbb{R}\,. \tag{3.2}$$

3.2 Proposition PROPERTIES OF DISTRIBUTION FUNCTIONS. *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$. Then*

(a) *$F(x)$ is nondecreasing;*

(b) *$F(x)$ is right-continuous;*

(c) *$F(x) \to 0$ as $x \to -\infty$;*

(d) *$F(x) \to 1$ as $x \to \infty$;*

(e) *$\mathsf{P}[X = x] = F(x) - F(x-)$;*

(f) *for any $x \in \mathbb{R}$ and $q \in (0,1)$,*

$$\{\,\mathsf{P}[X \le x] \ge q \text{ and } \mathsf{P}[X \ge x] \ge 1-q\,\} \Longleftrightarrow \{\,\mathsf{P}[X < x] \le q \text{ and } \mathsf{P}[X > x] \le 1-q\,\}\,.$$

3.3 Remark In view of Proposition **3.2**, the domain of a distribution function $F(x)$ can be extended to $\mathbb{R}$ $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, the extended real numbers, by setting

$$F(-\infty) = 0 \text{ and } F(\infty) = 1\,. \tag{3.3}$$

3.4 Proposition PROPERTIES OF SURVIVAL FUNCTIONS. *Let X be a real-valued random variable with distribution function $G(x) = \mathsf{P}[X \le x]$. Then*

(a) *$G(x)$ is nonincreasing;*

(b) *$G(x)$ is left-continuous;*

(c) *$G(x) \to 1$ as $x \to -\infty$;*

(d) *$G(x) \to 0$ as $x \to \infty$;*

(e) $\mathsf{P}[X = x] = G(x) - G(x+)$;

(f) $G(x) = 1 - F(x) + \mathsf{P}[S = x]$.

4. Quantile functions

4.1 Definition QUANTILE FUNCTION. *Let $F(x)$ be a distribution function. The quantile function associated with F is the generalized inverse of F, i.e.*

$$F^{-1}(q) \equiv F^{-}(q) = \inf\{x : F(x) \geq q\}\ ,\ 0 < q < 1\ . \tag{4.1}$$

4.2 Theorem PROPERTIES OF QUANTILE FUNCTIONS. *Let $F(x)$ be a distribution function. Then the following properties hold:*

(a) $F^{-1}(q) = \sup\{x : F(x) < q\}\ ,\ 0 < q < 1$;

(b) $F^{-1}(q)$ *is nondecreasing and left continuous;*

(c) $F(x) \geq q \Leftrightarrow x \geq F^{-1}(q)$, *for all* $x \in \mathbb{R}$ *and* $q \in (0, 1)$;

(d) $F(x) < q \Leftrightarrow x < F^{-1}(q)$, *for all* $x \in \mathbb{R}$ *and* $q \in (0, 1)$;

(e) $F[F^{-1}(q)-] \leq q \leq F[F^{-1}(q)]$, *for all* $q \in (0, 1)$;

(f) $F^{-1}[F(x)] \leq x \leq F^{-1}[F(x)+]$, *for all* $x \in \mathbb{R}$;

(g) *if* F *is continuous at* $x = F^{-1}(q)$, *then* $F[F^{-1}(q)] = q$;

(h) *if* F^{-1} *is continuous at* $q = F(x)$, *then* $F^{-1}[F(x)] = x$;

(i) *for* $q \in (0, 1)$, $F[F^{-1}(q)] = q \Leftrightarrow q \in F[\mathbb{R}]$;

(j) $F[F^{-1}(q)] = q$ *for all* $q \in (0, 1)$ $\Leftrightarrow (0, 1) \subseteq F[\mathbb{R}]$
$\Leftrightarrow F$ *is continuous*
$\Leftrightarrow F^{-1}$ *is strictly increasing*;

(k) *for any* $x \in \mathbb{R}$, $F^{-1}[F(x)] = x \Leftrightarrow F(x - \varepsilon) < F(x)$ *for all* $\varepsilon > 0$;

(l) $F^{-1}[F(x)] = x$ *for all* $x \in \mathbb{R}$ $\Leftrightarrow F$ *is strictly increasing*
$\Leftrightarrow F^{-1}$ *is continuous*;

(m) F *is continuous and strictly increasing* $\Leftrightarrow F^{-1}$ *is continuous and strictly increasing*;

(n) $F^{-1} \circ F \circ F^{-1} = F^{-1}$ *or, equivalently,*

$$F^{-1}\left(F\left[F^{-1}(q)\right]\right) = F^{-1}(q)\,,\ \text{for all } q \in (0, 1)\ ;$$

(o) $F \circ F^{-1} \circ F = F$ *or, equivalently,*

$$F\left(F^{-1}\left[F(x)\right]\right) = F(x)\,, \text{ for all } x \in \mathbb{R}\,.$$

4.3 Theorem CHARACTERIZATION OF DISTRIBUTIONS BY QUANTILE FUNCTIONS. *If $G(x)$ is a real-valued nondecreasing left continuous function with domain $(0, 1)$, there is a unique distribution function F such that $G = F^{-1}$.*

4.4 Theorem DIFFERENTIATION OF QUANTILE FUNCTIONS. *Let $F(x)$ be a distribution function. If F has a positive continuous $f(x)$ density f in a neighborhood of $F^{-1}(q_0)$, where $0 < q_0 < 1$, then the derivative $dF^{-1}(q)/dq$ exists at $q = q_0$ and*

$$\left.\frac{dF^{-1}(q)}{dq}\right|_{q_0} = \frac{1}{f\left(F^{-1}(q_0)\right)}\,. \tag{4.2}$$

4.5 Proposition *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \leq x]$ and survival function $G(x) = \mathsf{P}[X \geq x]$. Then, for any $q \in (0, 1)$,*

(a) $\mathsf{P}[X \leq F^{-1}(q)] \geq q$ *and* $\mathsf{P}[X \geq F^{-1}(q)] \geq 1 - q$;

(b) $\mathsf{P}[X < F^{-1}(q)] \leq q$ *and* $\mathsf{P}[X > F^{-1}(q)] \leq 1 - q$.

5. Quantile sets and generalized quantile functions

5.1 Notation *X is a random variable with distribution function $F_X(x) = \mathsf{P}[X \leq x]$. $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is the set of the extended real numbers.*

5.2 Definition QUANTILE OF RANDOM VARIABLE. *A* quantile *of order q (or a q-*quantile*) of the random variable X is any number $m_q \in \bar{\mathbb{R}}$ such that $\mathsf{P}[X \leq m_q] \geq q$ and $\mathsf{P}[X \geq m_q] \geq 1 - q$, where $0 \leq q \leq 1$. In particular, $m_{0.5}$ is a* median *of X, $m_{0.25}$ is a* first *(or* lower*)* quartile *of X, and $m_{0.75}$ is a* third *(or* upper*)* quartile *of X.*

5.3 Remark For $q = 0$, $m_q = -\infty$ always satisfies the quantile condition. If there is a finite number d_L such that $\mathsf{P}[X \leq d_L] = 0$, then any x such that $x \leq d_L$ is a quantile of order 0. Similarly, for $q = 1$, $m_q = \infty$ always satisfies the quantile condition. If there is a finite number d_U such that $\mathsf{P}[X \leq d_U] = U$, then any x such that $x \geq d_U$ is a quantile of order 1.

6. Distribution and quantile transformations

6.1 Notation *$U(0, 1)$ a uniform random variable on the interval $(0, 1)$.*

6.2 Theorem QUANTILES OF TRANSFORMED RANDOM VARIABLES. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathsf{P}[X \le x]$. If $g(x)$, $x \in \mathbb{R}$, is a nondecreasing left continuous function, then*

$$F_{g(X)}^{-1} = g\left(F_X^{-1}\right) \tag{6.1}$$

where $F_{g(X)}(x) = \mathsf{P}[g(X) \le x]$ and $F_{g(X)}^{-1}(q) = \inf\{x : F_{g(X)}(x) \ge q\}$, $0 < q < 1$.

6.3 Theorem PROPERTIES OF QUANTILE TRANSFORMATION. *Let $F(x)$ be a distribution function, and U a random variable with distribution $D(x)$ such that $D(0) = 0$ and $D(1) = 1$. If $X = F^{-1}(U)$, then, for all $x \in \mathbb{R}$,*

$$X \le x \Leftrightarrow F^{-1}(U) \le x \Leftrightarrow U \le F(x) \tag{6.2}$$

or, equivalently,

$$\mathbf{1}\{X \le x\} = 1\{F^{-1}(U) \le x\} = \mathbf{1}\{U \le F(x)\} \;, \tag{6.3}$$

and

$$\mathsf{P}[X \le x] = \mathsf{P}[F^{-1}(U) \le x] = \mathsf{P}[U \le F(x)] = D(F(x)) \;; \tag{6.4}$$

further,

$$\mathbf{1}\{X < x\} = 1\{F^{-1}(U) < x\} = \mathbf{1}\{U \le F(x-)\} \text{ with probability } 1 \tag{6.5}$$

and

$$\mathsf{P}[X < x] = \mathsf{P}[F^{-1}(U) < x] = \mathsf{P}[U \le F(x-)] \;. \tag{6.6}$$

In particular, if U follows a uniform distribution on the interval $(0, 1)$, i.e. $U \sim U(0, 1)$, the distribution function of X is F :

$$\mathsf{P}[X \le x] = \mathsf{P}[U \le F(x)] = F(x) \;. \tag{6.7}$$

6.4 Theorem PROPERTIES OF DISTRIBUTION TRANSFORMATION. *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$. Then the following properties hold:*

(*a*) $\mathsf{P}[F(X) \le u] \le u$, *for all* $u \in [0, 1]$;

(*b*) $\mathsf{P}[F(X) \le u] = u \Leftrightarrow u \in \mathrm{cl}\{F(\mathbb{R})\}$,
where $\mathrm{cl}\{F(\mathbb{R})\}$ *is the closure of the range of* F;

(*c*) $\mathsf{P}[F(X) \le F(x)] = \mathsf{P}[X \le x] = F(x)$, *for all* $x \in \mathbb{R}$;

(*d*) $F(X) \sim U(0, 1) \Leftrightarrow F$ *is continuous;*

(*e*) *for all* x, $\mathbf{1}\{F(X) \le F(x)\} = 1\{X \le x\}$ *with probability* 1;

(*f*) $F^{-1}(F(X)) = X$ *with probability* 1.

6.5 Theorem QUANTILES AND P-VALUES. *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$ and survival function $G(x) = \mathsf{P}[X \ge x]$. Then, for any $x \in \mathbb{R}$,*

$$G(x) \quad = \quad \mathsf{P}[G(X) \ge G(x)]$$

$$
\begin{aligned}
&= \mathsf{P}\big[X \geq F^{-1}\big((F(x) - p_F(x))^+\big)\big] \\
&= \mathsf{P}\big[X \geq F^{-1}\big((1 - G(x))^+\big)\big] \qquad (6.8)
\end{aligned}
$$

where $p_F(x) = \mathsf{P}[X = x] = F(x) - F(x-)$.

7. Multivariate generalizations

7.1 Notation CONDITIONAL DISTRIBUTION FUNCTIONS. *Let* $X = (X_1, \ldots, X_k)'$ *a* $k \times 1$ *random vector in* $\mathbb{R}^k$. *Then we denote as follows the following set of conditional distribution functions:*

$$
\begin{aligned}
F_{1|\cdot}(x_1) &= F_1(x_1) = \mathsf{P}[X_1 \leq x_1], \qquad (7.1) \\
F_{2|\cdot}(x_2|x_1) &= \mathsf{P}[X_2 \leq x_2 \,|\, X_1 = x_1], \\
&\vdots \\
F_{k|\cdot}(x_k \,|\, x_1, \ldots, x_{k-1}) &= \mathsf{P}[X_k \leq x_k \,|\, X_1 = x_1, \ldots, X_{k-1} = x_{k-1}].
\end{aligned}
$$

Further, we define the following transformations of $X_1, \ldots, X_k$:

$$
\begin{aligned}
Z_1 &= F_1(X_1), \qquad (7.2) \\
Z_2 &= F_{2|\cdot}(X_2 \,|\, X_1), \\
&\vdots \\
Z_k &= F_{k|\cdot}(X_k \,|\, X_1, \ldots, X_{k-1}).
\end{aligned}
$$

7.2 Theorem TRANSFORMATION TO *i.i.d.* $U(0,1)$ VARIABLES (ROSENBLATT). *Let* $X = (X_1, \ldots, X_k)'$ *be a* $k \times 1$ *random vector in* $\mathbb{R}^k$ *with an absolutely continuous distribution function* $F(x_1, \ldots, x_k) = \mathsf{P}[X_1 \leq x_1, \ldots, X_k \leq x_k]$. *Then the random variables* $Z_1, \ldots, Z_k$ *are independent and identically distributed according to a* $U(0, 1)$ *distribution.*

8. Proofs and additional references

1.4 - **1.5** Rudin (1976), Chapter 4, pp. 95-97, and Chung (1974), Section 1.1. For (a)-(b), see Phillips (1984), Sections 9.1 (p. 243) and 9.3 (p. 253).

1.6 - **1.8** Ramis, Deschamps, and Odoux (1982), Section 4.3.2, p.121.

1.9 Chung (1974), Section 1.1, p. 4.

1.11 Haaser and Sullivan (1991), Section 9.3; Riesz and Sz.-Nagy (1955/1990), Chapter 1.

2.3 (2.4) is proved by Reiss (1989, Appendix 1, Lemma A.1.1). (2.5) and (2.6) are also given by Gleser (1985, Lemma 1, p. 957).

2.4 Reiss (1989), Appendix 1, Lemma A.1.3.

2.5 Reiss (1989), Appendix 1, Lemma A.1.2.

3.2 (f) Lehmann and Casella (1998), Problem 1.7 (for the case $q = 1/2$).

4.2 (a) Williams (1991), Section 3.12 (p. 34).

7.2 See Rosenblatt (1952).

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