Basic asymptotic theory *

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First version: November 2011
Revised: December 2011
This version: December 2011
Compiled: December 12, 2011, 22:24

*This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

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Contents

1. Stochastic convergence

1.1. Basic definitions

1.1 Definition *Let $\{X_n = X_n(\omega) : n = 1,\ 2,\ ...\}$ a sequence of real r.v.'s defined on a probability space $(\Omega,\ \mathscr{A},\ P)$ and $X = X(\omega)$ another real r.v. defined on the same space.*

(a) *X_n converges* in probability *to X as $n \to \infty$ (denoted $X_n \xrightarrow{p} X$) iff*

$$\lim_{n\to\infty} P\left[|X_n - X| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0\,. \tag{1.1}$$

(b) *X_n converges* almost surely *to X as $n \to \infty$ (denoted $X_n \xrightarrow{a.s.} X$) iff*

$$P\left[\lim_{n\to\infty} X_n = X\right] = 1\,. \tag{1.2}$$

(c) *Suppose $E|X_n|^r < \infty\,, \forall n$, where $r > 0$. X_n converges* in mean of order r *to X (denoted $X_n \xrightarrow{r} X$) iff*

$$\lim_{n\to\infty} E\left[|X_n - X|^r\right] = 0\,. \tag{1.3}$$

In this case, we also say that X_n converges to X in L_r. If $r = 2$, we say X_n converges to X in quadratic mean $(q.m.)$.

(d) *Let $F_n(x)$ and $F(x)$ be the distribution functions of X_n and X respectively. X_n converges* in law *(or* in distribution*) to X as $n \to \infty$ (denoted $X_n \xrightarrow{L} X$) iff*

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ at all continuity points of } F(x)\,. \tag{1.4}$$

An important specila case of the above concepts is the one where X is a fixed real constant c.

1.2 Definition *Let $\{X_n = X_n(\omega) : n = 1,\ 2,\ ...\}$ a sequence of real r.v.'s defined on a probability space $(\Omega,\ \mathscr{A},\ P)$ and c a real constant.*

(a) *X_n converges* in probability *to X as $n \to \infty$ (denoted $X_n \xrightarrow{p} c$) iff*

$$\lim_{n\to\infty} P\left[|X_n - c| > \varepsilon\right] = 0\,,\ \forall\varepsilon > 0\,. \tag{1.5}$$

(b) *X_n converges* almost surely *to c as $n \to \infty$ (denoted $X_n \xrightarrow{a.s.} c$) iff*

$$P\left[\lim_{n\to\infty} X_n = c\right] = 1\,. \tag{1.6}$$

(c) *Suppose $E|X_n|^r < \infty\,, \forall n$, where $r > 0$. X_n converges* in mean of order r *to c (denoted $X_n \xrightarrow{r} c$) iff*

$$\lim_{n\to\infty} E\left[|X_n - c|^r\right] = 0\,. \tag{1.7}$$

In this case, we also say that X_n converges to c in L_r. If $r = 2$, we say X_n converges to c in quadratic mean $(q.m.)$.

1.3 Proposition UNICITY OF PROBABILITY LIMIT. *Let $\{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathscr{A}, P)$, and let X and Y be two real r.v.'s defined on the same probability space. Then*

$$X_n \xrightarrow{p} X \text{ and } X_n \xrightarrow{p} Y \Rightarrow P[X \neq Y] = 0\,. \tag{1.8}$$

1.2. Relations between convergence concepts

1.4 Assumption *Let $\{X_n\} \equiv \{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathscr{A}, P)$ and X another real r.v. defined on the same space.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.5 Proposition RELATIONS BETWEEN CONVERGENCE CONCEPTS.

(a) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X$.

(b) $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{s} X$ *for all s such that* $0 < s \leq r \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X$.

1.6 Remark In general, the implications in **1.5** (b) and (c) cannot be reversed.

1.3. Convergence of expectations and functions of random variables

1.1 Assumption *Let $\{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s, X a real r.v. and $g : \mathbb{R} \to \mathbb{R}$ a function such that $g(X)$ and $g(X_n)$, $n = 1, 2, ...$, are real r.v.'s.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.2 Proposition *Let $g : \mathbb{R} \to \mathbb{R}$ a continuous function everywhere on $\mathbb{R}$, except possibly in a set $A \subseteq \mathbb{R}$, and let X a r.v. such that $P[X \in A] = 0$. Then*

(a) $X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X)$;

(b) $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$;

$(c)\;\; X_n \xrightarrow{L} X \Rightarrow g(X_n) \xrightarrow{L} g(X)\,.$

1.3 Proposition *Let* $\{X_n\}$ *and* $\{Y_n\}$ *two sequences of random variables. Then*

$(a)\;\; X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \Rightarrow X_n + Y_n \xrightarrow{p} X + Y\,;$

$(b)\;\; X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y \Rightarrow X_n + Y_n \xrightarrow{a.s.} X + Y\,;$

$(c)\;\; X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \Rightarrow g(X_n, Y_n) \xrightarrow{p} g(X, Y)$ *for any continuous function* $g(x,\, y)\,.$

1.4 Proposition *Let* $\{X_n\}$ *and* $\{Y_n\}$ *two sequences of r.v.'s such that* $X_n \xrightarrow{L} X$ *and* $Y_n \xrightarrow{p} c$, *where* X *is a r.v. and* c *is a real constant* $(-\infty < c < +\infty)$. *Then*

$(a)\;\; X_n + Y_n \xrightarrow{L} X + c\,;$

$(b)\;\; X_n Y_n \xrightarrow{L} Xc\,;$

$(c)\;\; X_n / Y_n \xrightarrow{L} X/c$ *if* $c \neq 0\,;$

$(d)\;\; (X_n,\, Y_n) \xrightarrow{L} (X,\, c)\,.$

1.5 Proposition *Let* $\{X_n\}$ *and* $\{Y_n\}$ *two sequences of r.v.'s such that* $X_n - Y_n \xrightarrow{p} 0$ *and* $Y_n \xrightarrow{L} Y$, *and let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Then*

$(a)\;\; X_n \xrightarrow{L} Y\,;$

$(b)\;\; g(X_n) - g(Y_n) \xrightarrow{p} 0\,;$

$(c)\;\; g(X_n) \xrightarrow{L} g(Y)\,.$

1.4. Random series

1.6 Definition *Let* $\{X_t : t \in \mathbb{N}\}$ *be a real-valued stochastic process and consider the series* $\sum_{t=1}^{\infty} X_t$.

1.7 Definition *We say* $\sum_{t=1}^{\infty} X_t$ *converges (according to given mode of convergence) iff there exists a real r.v.* Y *such that*

$$\sum_{t=1}^{N} X_t \underset{N\to\infty}{\longrightarrow} Y \text{ (according to the same mode of convergence)}\,.$$

1.8 Remark The mode of convergence : a.s., in probability or in mean of order r .

2. Laws of large numbers

2.1 Proposition *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of r.v.'s such that* $X_t \in L_2$ *and* $Cov(X_s,\ X_t) = 0$ *for* $s \neq t$*, and let* $\mu_t = E(X_t)$*. Then*

(*a*) $\sum_{n=1}^{\infty} \frac{1}{n^2} Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{2} 0$ (*Chebychev law*)

where $\bar{X}_n = \sum_{t=1}^{n} X_t/n$ *and* $\bar{\mu}_n = \sum_{t=1}^{n} \mu_t/n$, *and*

(*b*) $\sum_{n=1}^{\infty} \left(\frac{\log n}{n}\right)^2 Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0$.

In particular, if $Var(X_t) = \sigma^2 < \infty$ *and* $E(X_t) = \mu$ *for all* t*, then*

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{a.s.} \mu \text{ and } \frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{2} \mu \,.$$

2.2 Theorem KHINTCHINE WEAK LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent and identically distributed r.v.'s whose mean* $E(X_t)$ *exists. Then*

$$E(X_t) = \mu \Rightarrow \bar{X}_n \xrightarrow[n\to\infty]{p} \mu \,.$$

2.3 Theorem FIRST KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent r.v.'s such that* $E(X_t) = \mu_t$ *and* $Var(X_t) = \sigma_t^2$ *exist for all* t*. Then*

$$\sum_{n=1}^{\infty} (\sigma_n/\ n)^2 < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0 \,.$$

2.4 Theorem SECOND KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let* $\{X_t\}_{t=1}^{\infty}$ *a sequence of independent and identically distributed r.v.'s. Then*

$$E(X_t) \text{ exists and is equal to } \mu \Leftrightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0 \,.$$

3. Central limit theorems

3.1 Theorem LINDEBERG-LÉVY CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s in L_2 such that $E(X_t) = \mu$ and $Var(X_t) = \sigma^2 > 0$. Then*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(X_t - \mu)/\ \sigma = \sqrt{n}(\bar{X}_t - \mu)/\ \sigma \xrightarrow[n\to\infty]{L} Z$$

where $Z \sim N(0,1)$.

3.2 Theorem LIAPUNOV CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_3 such that $E(X_t) = \mu_t$, $Var(X_t) = \sigma_t^2 \neq 0$, $E[|X_t - \mu_t|^3] = \beta_t$ for all t. Moreover,*

$$B_n = \left(\sum_{t=1}^{n}\beta_t\right)^{1/3}, C_n = \left(\sum_{t=1}^{n}\sigma_t^2\right)^{1/2}.$$

If $\lim_{n\to\infty}(B_n/\ C_n) = 0$, then

$$\Sigma_{t=1}^{n}(X_t - \mu_t)/\ C_n = \sqrt{n}(\bar{X}_t - \bar{\mu}_n)/(C_n/\sqrt{n})\sigma \xrightarrow[n\to\infty]{L} Z$$

where $Z \sim N(0,1)$.

3.3 Theorem LINDEBERG-FELLER CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_2 such that*

$$P[X_t \leq x] = G_t(x)\ ,\ E(X_t) = \mu_t\ ,\ Var(X_t) = \sigma_t^2 \neq 0\ ,$$

for all t. Then

$$\sum_{t=1}^{n}(X_t - \mu_t)/\ C_n \xrightarrow[n\to\infty]{L} Z \text{ and } \lim_{n\to\infty}\max_{1\leq t\leq n}(\sigma_t/\ C_n) = 0$$

iff

$$\lim_{n\to\infty}\frac{1}{C_n^2}\sum_{t=1}^{n}\int_{|x-\mu_t|>\varepsilon C_n}(x-\mu_t)^2 dG_t(x) = 0, \forall\varepsilon > 0\,.$$

3.1. Extension to random vectors

3.1 Definition STOCHASTIC CONVERGENCE FOR VECTORS. *Let $\{X_n\}_{n=1}^{\infty}$ a sequence of vectors of dimension k,*

$$X_n = (X_{1n}, X_{2n}, ..., X_{kn})', n = 1, 2, ...$$

whose components are real random variables all defined on the same probability space (Ω, Q, P), *and*

$$X = (X_1, X_2, \ldots, X_k)'$$

another random vector of dimension k *whose components are defined on the same space.*

(a) *We say* X_n *converges to* X *in probability* $($*almost surely, in mean of order* $r)$ *as* $n \to \infty$ *if each component of* X_n *converges to the corresponding component of* X *in probability* $($*almost surely, in mean of order* $r)$ *as* $n \to \infty$. *Depending on the case considered, we then write* $X_n \xrightarrow{p} X$, $X_n \xrightarrow{a.s.} X$ *or* $X_n \xrightarrow{r} X$.

(b) *We say* X_n *converges in law to* X $(X_n \xrightarrow{L} X)$ *iff*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ at all continuity points of } F_X(x)\,.$$

where $x = (x_1, x_2, \ldots, x_k)' \in \mathbb{R}^k$,

$$F_{X_n}(x) = P[X_{1n} \leq x_1, \ldots, X_{kn} \leq x_k]\,,\; n = 1, 2, \ldots$$

and

$$F_X(x) = P[X_1 \leq x_1, \ldots, X_k \leq x_k]\,.$$

3.2 Theorem UNIVARIATE CHARACTERIZATION OF CONVERGENCE IN LAW FOR A SEQUENCE OF VECTORS.. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random vectors of dimension* $k \times 1$ *and let* X *be another random vector of dimension* $k \times 1$. *Then*

$$X_n \underset{n\to\infty}{\xrightarrow{L}} X \Leftrightarrow \lambda' X_n \underset{n\to\infty}{\xrightarrow{L}} \lambda' X\,, \forall \lambda \in \mathbb{R}^k\,.$$

In particular, if $X \sim N[\mu, \Sigma]$,

$$X_n \underset{n\to\infty}{\xrightarrow{L}} N[\mu, \Sigma] \Leftrightarrow \lambda' X_n \underset{n\to\infty}{\xrightarrow{L}} N[\lambda'\mu, \lambda'\Sigma\lambda], \forall \lambda \in \mathbb{R}^k\,.$$

3.2. Proofs and additional references

Proofs and further discussions of the results presented above may be found the following references: Rao (1973), Lukacs (1975), Stout (1974), Loève (1977).

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