# Prediction and regression [*]

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First version: May 1999
Revised: February 2003, September 2011
This version: September 2011
Compiled: September 12, 2011, 12:11

[*] This work was supported by the William Dow Chair in Political Economy (McGill University), the Canada Research Chair Program (Chair in Econometrics, Université de Montréal), the Bank of Canada (Research Fellowship), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Institut de finance mathématique de Montréal (IFM2), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

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# List of Definitions, Assumptions, Propositions and Theorems

# 1. Optimal mean square prediction

Let $Y,\ X_1, \ldots, X_k$ be real random variables in $L^2$, and $X = (X_1, \ldots, X_k)'$. We wish to find a function

$$g(X) = g(X_1, \ldots, X_k)$$

such that

$$\mathsf{E}\big([Y - g(X)]^2\big) \text{ is minimal.}$$

Given the mean square criterion, we also restrict $g(X)$ to be in $L^2$ :

$$\mathsf{E}\left[g(X)^2\right] < \infty.$$

Then it is easy to see that the optimal solution to this problem is

$$g(X) = M(X)$$

where

$$M(X) = \mathsf{E}(Y \mid X)\ .$$

In general, $M(X)$ is a nonlinear function of $X$. The optimality of $M(X)$ can easily be shown on observing that :

$$\begin{aligned}
\mathsf{E}\left\{[Y - g(X)]^2\right\} &= \mathsf{E}\left\{[Y - \mathsf{E}(Y \mid X) + \mathsf{E}(Y \mid X) - g(X)]^2\right\} \\
&= \mathsf{E}\Big\{[Y - \mathsf{E}(Y \mid X)]^2 + [\mathsf{E}(Y \mid X) - g(X)]^2 \\
&\quad + 2[Y - \mathsf{E}(Y \mid X)][\mathsf{E}(Y \mid X) - g(X)]\Big\} \\
&= \mathsf{E}\left\{[Y - \mathsf{E}(Y \mid X)]^2\right\} + \mathsf{E}\left\{[\mathsf{E}(Y \mid X) - g(X)]^2\right\} \\
&\quad + 2E\{[\mathsf{E}(Y \mid X) - g(X)]\, \mathsf{E}[Y - \mathsf{E}(Y \mid X) \mid X]\} \\
&= \mathsf{E}\left\{[Y - \mathsf{E}(Y \mid X)]^2\right\} + \mathsf{E}\left\{[\mathsf{E}(Y \mid X) - g(X)]^2\right\}
\end{aligned}$$

from which it follows that the optimal solution is

$$g(X) = \mathsf{E}(Y \mid X)\ .$$

The set of random variables

$$M_0 = \left\{Z : Z = g(X) \text{ is a random variable and } \mathsf{E}\left(Z^2\right) < \infty\right\}$$

is a closed subspace of $L^2$. $M(X) = \mathsf{E}(Y \mid X)$ can be interpreted as the projection of $Y$ on $M_0$ :

$$\mathsf{E}(Y \mid X) = P_{M_0} Y.$$

## 2. Properties of conditional expectations

Let

$$\begin{aligned} Y &= (Y_1, \ldots, Y_q)', \\ Z &= (Z_1, \ldots, Z_q)', \\ X &= (X_1, \ldots, X_k) \end{aligned}$$

be random vectors whose components are all in $L^2$. By definition,

$$\mathsf{E}(Y \mid X) = \begin{bmatrix} \mathsf{E}(Y_1 \mid X) \\ \mathsf{E}(Y_2 \mid X) \\ \vdots \\ \mathsf{E}(Y_q \mid X) \end{bmatrix}$$

and similarly for $\mathsf{E}(Z \mid X)$.

Let $L^2(X)$ be the set of random variables $W$ such that $W = g(X)$ and $\mathsf{E}\left(W^2\right) < \infty$.

**2.1 Proposition** LINEARITY. *Let $A$ an $m \times q$ fixed matrix and $b$ an $m \times 1$ fixed vector. Then*

$$\begin{aligned} \mathsf{E}(AY + b \mid X) &= AE(Y \mid X) + b, \\ \mathsf{E}(Y + Z \mid X) &= \mathsf{E}(Y \mid X) + \mathsf{E}(Z \mid X). \end{aligned}$$

**2.2 Proposition** POSITIVITY. *If $Y_i \geq 0$, for $i = 1, \ldots, q$, then*

$$\mathsf{E}(Y_i \mid X) \geq 0, \quad \text{for} \quad i = 1, \ldots, q.$$

**2.3 Proposition** MONOTONICITY. *If $Y_i \geq Z_i$, for $i = 1, \ldots, q$, then*

$$\mathsf{E}(Y_i \mid X) \geq \mathsf{E}(Z_i \mid X), \quad \text{for} \quad i = 1, \ldots, q.$$

**2.4 Proposition** INVARIANCE.

$$\begin{aligned} \mathsf{E}(Y \mid X) = Y \quad &\Leftrightarrow \quad Y \text{ is a function of } X \\ &\Leftrightarrow \quad \text{there is a function } g(x) \text{ such that } Y = g(X) \\ &\qquad \text{with probability } 1. \end{aligned}$$

**2.5 Proposition** ORTHOGONALITY. *If $g_1(X) \in L^2$ and $g_2(Y) \in L^2$, then*

$$\mathsf{E}\{g_1(X)[g_2(Y) - \mathsf{E}(g_2(Y) \mid X)]\} = 0.$$

**2.6 Proposition** ITERATED CONDITIONINGS LAW. *If $W$ is a random vector such that*

$$L^2(W) \subseteq L^2(X),$$

*then*

$$\begin{aligned} \mathsf{E}\left[\mathsf{E}\left(Y \mid X\right) \mid W\right] &= \mathsf{E}\left[\mathsf{E}\left(Y \mid W\right) \mid X\right] \\ &= \mathsf{E}\left(Y \mid W\right). \end{aligned}$$

**2.7 Proposition** MEAN SQUARE OPTIMALITY.

$$\mathsf{E}\left[\left(Y_i - \mathsf{E}\left(Y_i \mid X\right)\right)^2\right] = \min_{g_i(X) \in L^2(X)} \mathsf{E}\left[\left(Y_i - g_i\left(X\right)\right)^2\right],\ i = 1, \ldots, q.$$

**2.8 Proposition** CHARACTERIZATION OF OPTIMALITY BY ORTHOGONALITY. *For any* $i = 1, \ldots, q$*,*

$$h_i\left(X\right) = \mathsf{E}\left(Y_i \mid X\right) \Leftrightarrow \mathsf{E}\left[g\left(X\right)\left(Y_i - h_i\left(X\right)\right)\right] = 0\,,\ \forall g\left(X\right) \in L^2\left(X\right).$$

**2.9 Definition** CONDITIONAL COVARIANCE. *The conditional covariance matrix of $Y$ given $X$ is the matrix*

$$\mathsf{V}\left(Y \mid X\right) = \mathsf{E}\left[\left(Y - \mathsf{E}\left(Y \mid X\right)\right)\left(Y - \mathsf{E}\left(Y \mid X\right)\right)' \mid X\right].$$

If we define

$$\varepsilon\left(X\right) = Y - \mathsf{E}\left(Y \mid X\right),$$

we see easily that

$$\mathsf{V}\left[\varepsilon\left(X\right)\right] = \mathsf{E}\left[\mathsf{V}\left(Y \mid X\right)\right].$$

We can then write

$$Y = \mathsf{E}\left(Y \mid X\right) + \varepsilon\left(X\right)$$

where $\mathsf{E}\left(Y \mid X\right)$ and $\varepsilon\left(X\right)$ are uncorrelated.

**2.10 Proposition** VARIANCE DECOMPOSITION.

$$\begin{aligned} \mathsf{V}\left(Y\right) &= \mathsf{V}\left[\mathsf{E}\left(Y \mid X\right)\right] + \mathsf{V}\left[\varepsilon\left(X\right)\right] \\ &= \mathsf{V}\left[\mathsf{E}\left(Y \mid X\right)\right] + \mathsf{E}\left[\mathsf{V}\left(Y \mid X\right)\right]. \end{aligned}$$

# 3. Linear regression

Consider again the setup of Section 1. We now study the problem of finding a function of the form

$$\begin{aligned} L\left(X\right) &= b_0 + b_1X_1 + \cdots + b_kX_k \\ &= \sum_{i=0}^{k} b_iX_i = b'x \end{aligned}$$

where

$$X_0 = 1,\ b = \left(b_0, b_1, \ldots, b_k\right)' \tag{3.1}$$

$$x = (X_0, X_1, \ldots, X_k)', \tag{3.2}$$

such that the mean square prediction error

$$\mathsf{E}\left\{[Y - L(X)]^2\right\} = \mathsf{E}\left[\left(Y - b'x\right)^2\right]$$

is minimal. In other words, we wish to minimize (with respect to $b$) the function

$$\begin{aligned} S(b) &= \mathsf{E}\left\{\left[Y - b'x\right]^2\right\} \\ &= \mathsf{E}\left(Y^2\right) - 2b'\mathsf{E}(xY) + b'\mathsf{E}\left(xx'\right)b\,. \end{aligned}$$

It is easy to see that the optimal value of $b$ must satisfy the equation

$$\mathsf{E}\left[x\left(Y - b'x\right)\right] = 0$$

or

$$\mathsf{E}\left(xx'\right)b = \mathsf{E}(xY)\ .$$

If we write

$$b = \begin{pmatrix} \beta_0 \\ \gamma \end{pmatrix},\ \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix},\ X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix},$$

we see that

$$\begin{bmatrix} 1 & \mathsf{E}(X)' \\ \mathsf{E}(X) & \mathsf{E}(XX') \end{bmatrix} \begin{bmatrix} \beta_0 \\ \gamma \end{bmatrix} = \begin{bmatrix} \mathsf{E}(Y) \\ \mathsf{E}(XY) \end{bmatrix},$$

hence

$$\beta_0 + \mathsf{E}(X)'\gamma = \mathsf{E}(Y) \tag{3.3}$$

$$\mathsf{E}(Y)\beta_0 + \mathsf{E}\left(XX'\right)\gamma = \mathsf{E}(XY) \tag{3.4}$$

and

$$\beta_0 = \mathsf{E}(Y) - \mathsf{E}(X)'\gamma\,.$$

Further, by the basic properties of the expectation operator,

$$\begin{aligned} \mathsf{E}\left(XX'\right) &= \mathsf{V}(X) + \mathsf{E}(X)\mathsf{E}(X)'\ , \\ \mathsf{E}(XY) &= \mathsf{C}(X,Y) + \mathsf{E}(X)\mathsf{E}(Y) \end{aligned}$$

where

$$\mathsf{V}(X) = \mathsf{E}\left\{\mathsf{E}[X - \mathsf{E}(X)][X - \mathsf{E}(X)]'\right\}\ , \tag{3.5}$$

$$\mathsf{C}(X,Y) = \mathsf{E}\left\{[X - \mathsf{E}(X)][Y - \mathsf{E}(Y)]'\right\}\ . \tag{3.6}$$

By the equations (3.3)-(3.6), we then see easily that

$$\begin{aligned} \mathsf{E}(X)\beta_0 + \mathsf{E}(X)\mathsf{E}(X)'\gamma &= \mathsf{E}(X)\mathsf{E}(Y)\ , \\ \mathsf{E}(X)\beta_0 + \mathsf{V}(X)\gamma + \mathsf{E}(X)\mathsf{E}(X)'\gamma &= \mathsf{C}(X,Y) + \mathsf{E}(X)\mathsf{E}(Y) \end{aligned}$$

hence

$$\mathsf{V}(X)\gamma = \mathsf{C}(X,Y)\ .$$

Thus,

$$\begin{aligned} \beta_0 &= \mathsf{E}(Y) - \mathsf{E}(X)'\gamma\,, & (3.7) \\ \mathsf{V}(X)\gamma &= \mathsf{C}(X,Y)\ . & (3.8) \end{aligned}$$

The function

$$L(X) = \beta_0 + X'\gamma$$

is called the

*linear regression of $X$ on $Y$*

or the

*affine projection of $Y$ on $X$.* (3.9)

We write

$$L(X) = P_L(Y \mid X) = \beta_0 + X'\gamma$$

where $\beta_0$ and $\gamma$ are any solution of the normal equations:

$$\begin{aligned} \mathsf{V}(X)\gamma &= \mathsf{C}(X,Y)\ , \\ \beta_0 &= \mathsf{E}(Y) - \mathsf{E}(X)'\gamma\,. \end{aligned}$$

If we denote by

$$\varepsilon = Y - P_L(Y \mid X)$$

the prediction error, we see easily that:

$$\begin{aligned} \mathsf{E}(\varepsilon) &= 0\,, \\ \mathsf{C}(X,\varepsilon) &= 0\,. \end{aligned}$$

In the language of Hilbert space theory, we can also write

$$L(X) = P_M Y = P_L(Y \mid X)$$

where

$$M = \overline{sp}\{1, X\} = \overline{sp}\{1, X_1,\ \ldots,\ X_k\}\ .$$

If

$$\det[\mathsf{V}(X)] \neq 0\ ,$$

the optimal coefficients $\beta_0$ and $\gamma$ are uniquely defined :

$$\gamma = \mathsf{V}(X)^{-1}\,\mathsf{C}(X,Y)\,,\;\; \beta_0 = \mathsf{E}(Y) - \mathsf{E}(X)'\gamma\,.$$

# 4. Bibliographic notes

On the properties of conditional expectations, see Gouriéroux and Monfort (1995, Appendix B) and Williams (1991).

## References

GOURIÉROUX, C., AND A. MONFORT (1995): *Statistics and Econometric Models, Volumes One and Two*. Cambridge University Press, Cambridge, U.K., Translated by Quang Vuong.

WILLIAMS, D. (1991): *Probability with Martingales*. Cambridge University Press, Cambridge, U.K.