Solution to Econ 763 Assignment 1 (Winter 2017)
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Grading scheme: 5+5 for Problem 1; 4×5 for Problem 2; $5 + 5 + 10$ for Problem 3; and $30 + 20$ for Problem 4

Problem 1

(a) Define the notion of **probability space**.

(b) Define the notion of real-valued **stochastic process** on a probability space.

Answer.
(a). A probability space is a triplet $(\Omega, \mathcal{A}, P)$ satisfying:

(1) Ω is a set of possible outcomes;

(2) $\mathcal{A}$ is a σ**-algebra** on Ω, meaning

  (i) $\emptyset$ and Ω are in $\mathcal{A}$,
  (ii) if $A \in \mathcal{A}$ and $\Omega \backslash A$ is in $\mathcal{A}$,
  (iii) if $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets of Ω such that each $A_n \in \mathcal{A}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$;

(3) $P : \mathcal{A} \to \mathbb{R}$ is a **probability measure**, meaning

  (i) $P(A) \geq 0$ for any $A \in \mathcal{A}$,
  (ii) $P(\emptyset) = 0$ and $P(\Omega) = 1$,
  (iii) if $\{A_n\}_{n=1}^{\infty}$ is a sequence of *disjoint* subsets of Ω such that each $A_n \in \mathcal{A}$, then $P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$.

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A subset A of Ω is called an **event**. If $A \in \mathcal{A}$, then A is **measurable** (by $\mathcal{A}$ and P) else A is not measurable.

(b). Let T be a non-empty set. A **stochastic process on** T is a collection of random variables $X_t : \Omega \to \mathbb{R}$ such that to each element $t \in T$, we associate a random variable X_t. The process can then be written as $\{X_t : t \in T\}$. If $T = \mathbb{R}$ (real numbers), we have a process in **continuous time**. If $T = \mathbb{Z}$ or $T \subset \mathbb{Z}$, we have a **discrete time process**. $\square$

Problem 2

Answer by TRUE, FALSE or UNCERTAIN to each one of the following statements. Justify briefly your answer.

(1) Any strictly stationary process is in L_2.

FALSE. Suppose $\{y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. random variables each of which follows a t-distribution with 2 degrees of freedom. The i.i.d. assumption implies strict stationarity. Yet, because a t-distribution with v degrees of freedom does not have moments of order v or above, we infer that each y_t does not possess a finite variance. In particular, because $E(y_t) = 0$, we have $\text{Var}(y_t) = E(y_t^2) = +\infty$. $\square$

(2) Any strictly stationary process is also second-order stationary.

FALSE. The process $\{y_t\}_{t=1}^{\infty}$ described in (1) is strictly stationary because the variance of each y_t is infinite. $\square$

(3) Any stationary process of order 3 is also stationary of order 2.

TRUE. Suppose that $\{X_t\}_{t \in T}$ satisfies stationarity of order 3. That is,

(i) $E(|X_t|^3) < \infty$, for all $t \in T$,

(ii) $E[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}] = E[X_{t_1+k}^{m_1} \dots X_{t_n+k}^{m_n}]$ for any $k \geq 0$, any subset $\{t_1, \dots, t_n\} \in T^n$ and all the non-negative integers $m_1, \dots, m_n$ s.t. $m_1 + m_2 + \dots + m_n \leq 3$.

By the Jensen's inequality with $Z = |X_t|^2 \geq 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $g(z) = z^{3/2}$, we have

$$g[E(Z)] \leq E(g(Z)) \iff [E(|X_t|^2)]^{3/2} \leq E(|X_t|^3) < \infty.$$

The last inequality above is due to (i). We may infer that $E(|X_t|^2) < \infty$. Next, we use condition (ii) above with $t_1 = t$ and $k = s - t$ (s and t are some integers in T), $n = 1$, and $m_1 = 1$ to obtain:

$$E(X_t^1) = E(X_{t+k}^1) \iff E(X_t) = E(X_s).$$

Similarly, we can use condition (ii) with $n = 2$, $t_1 = s$, $t_2 = t$ (again, s and t are some integers in T), $k \geq 0$, and $m_1 = m_2 = 2$ (which satisfies $m_1 + m_2 \leq 3$) to obtain $E(X_s X_t) = E(X_{s+k} X_{t+k})$, which in turn implies

$$\begin{aligned}\mathrm{Cov}(X_s, X_t) &= E(X_s X_t) - E(X_s)E(X_t) \\ &= E(X_{s+k} E_{t+k}) - E(X_{s+k})E(X_{t+k}) \\ &= \mathrm{Cov}(X_{s+k}, X_{t+k}).\end{aligned}$$

We have thus verify stationarity of second order. □

(4) Any asymptotically stationary process of order 3 is also asymptotically stationary process of order 2.

TRUE. Suppose that $\{X_t\}_{t \in T}$ is asymptotically stationary of order 3. This means that

(i) there exists an integer N s.t. $E(|X_t|^3) < \infty$, for $t \geq N$, and

(ii)
$$\lim_{t_1 \to \infty} E(X_{t_1}^{m_1} X_{t_1+\Delta_2}^{m_2} \ldots X_{t_1+\Delta_n}^{m_n}) = \lim_{t_1 \to \infty} E(X_{t_1+k}^{m_1} X_{t_1+\Delta_2+k}^{m_2} \ldots X_{t_1+\Delta_n+k}^{m_n})$$
for any $k \geq 0$, $t_1 \in T$, all the positive integers $\Delta_2, \Delta_3, \ldots, \Delta_n$ s.t. $\Delta_2 < \Delta_3 < \ldots < \Delta_n$, and the non-negative integers $m_1, \ldots, m_n$ s.t. $m_1 + m_2 + \cdots + m_n \leq 3$.

By the Jensen's inequality, condition (i) above implies that for the same N above, $E(|X_t|^2)$ is finite for all $t \geq N$. Moreover, if $m_1 + m_2 + \cdots + m_n \leq 2$, then it's trivial that $m_1 + m_2 + \cdots + m_n \leq 3$. Thus, (ii) implies the analogous condition for asymptotic stationarity of order 2. □

(5) A white noise is a stationary process of order 4.

FALSE. Suppose that $\{X_t\}_{t \geq 1}$ is a sequence of i.i.d. random variables, each of which follows a t distribution with 4 degrees of freedom. Such a t distribution has finite moments of order 3 and below, and does not have moments of order 4 and above. Together with the i.i.d. assumption, this implies that $\{X_t\}_{t \geq 1}$ is a white noise process. But because $E(|X_t|^4) = \infty$, the process is **not** stationary of order 4. □

Problem 3

Let $\gamma(k)$ the autocovariance function of second-order stationary process on the integers. Prove that:

(a) $\gamma(0) = \mathrm{Var}(X_t)$ and $\gamma(k) = \gamma(-k)$, for all $k \in \mathbb{Z}$;

(b) $|\gamma(k)| \leq \gamma(0)$, $\forall k \in \mathbb{Z}$;

(c) the function $\gamma(k)$ is positive semi-definite.

Proof.
(a). By definition, we have $\mathrm{Cov}(X_s, X_t) = \gamma(t-s)$ for all $s, t \in T$. By the second-order stationarity, the autocovariance function γ is well-defined. In particular,

$$\gamma(0) = \mathrm{Cov}(X_t, X_t) = \mathrm{Var}(X_t)$$

and (s below is any integer)

$$\gamma(k) = \mathrm{Cov}(X_s, X_{s+k}) = \mathrm{Cov}(X_{s+k}, X_s) = \gamma(s-(s+k)) = \gamma(-k).$$

(b). That $|\gamma(k)| \leq \gamma(0)$ is a consequence of the Cauchy-Schwarz inequality. We provide here a direct proof for completeness. With a fixed $k \in \mathbb{Z}$ and any $z \in \mathbb{R}$, we have

$$0 \leq \mathrm{Var}(X_s - zX_{s+k}) = \mathrm{Var}(X_s) - 2z\,\mathrm{Cov}(X_s, X_{s+k}) + z^2\,\mathrm{Var}(X_{s+k}).$$

The rightmost expression above is a quadratic polynomial in z so it is nonnegative for all real z iff the discriminant is nonpositive:

$$0 \leq [-2\,\mathrm{Cov}(X_s, X_{s+k})]^2 - 4\,\mathrm{Var}(X_s)\,\mathrm{Var}(X_{s+k}) \iff \gamma(k)^2 \leq \gamma(0)^2$$

from which $|\gamma(k)| \leq |\gamma(0)| = \gamma(0)$ follows immediately.
(c). We have to show that: for any positive integer N and for all vectors $a = (a_1, \ldots, a_N)' \in \mathbb{R}^N$ and $\tau = (t_1, \ldots, t_N)' \in T^N$, it holds that $\sum_{i=1}^N \sum_{j=1}^N a_i a_j \gamma(t_i - t_j) \geq 0$. This follows by considering $Z \equiv a_1 X_{t_1} + \cdots a_N X_{t_N}$. We have

$$0 \leq \mathrm{Var}(Z) = \mathrm{Cov}(Z, Z) = \mathrm{Cov}\left(\sum_{i=1}^N a_i X_{t_i}, \sum_{j=1}^N a_j X_{t_j}\right)$$

$$= \sum_{i=1}^N \sum_{j=1}^N \mathrm{Cov}(a_i X_{t_i}, a_j X_{t_j}) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j\,\mathrm{Cov}(X_{t_i}, X_{t_j}) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \gamma(t_i - t_j).$$

□

Problem 4

Consider a process that follows the following model:

$$X_t = \sum_{j=1}^m [A_j \cos(v_j t) + B_j \sin(v_j t)], \quad t \in \mathbb{Z},$$

where $v_1, \ldots, v_m$ are distinct constants on the interval $[0, 2\pi)$ and A_j, B_j ($j = 1, \ldots, m$) are random variables in L_2 such that

$$E(A_j) = E(B_j) = 0, \quad E(A_j^2) = E(B_j^2) = \sigma_j^2, \quad j = 1, \ldots, m,$$
$$E(A_j A_k) = E(B_j B_k) = 0 \quad j \neq k,$$
$$E(A_j B_k) = 0 \quad \forall j, k.$$

(a) Show that this process is second-order stationary.

(b) For the case where $m = 1$, show that this process is deterministic.

Proof.
(a). We have to check

(1) $E(X_t^2) < \infty$, $\forall t \in \mathbb{Z}$,

(2) $E(X_t) = E(X_s)$ for all $s, t \in T$,

(3) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+k}, X_{t+k})$ for all $s, t \in T$, $\forall k \geq 0$.

Showing (2) is the easiest:

$$E(X_t) = \sum_{j=1}^{m} (\cos(v_j t) E(A_j) + \sin(v_j t) E(B_j)) = 0$$

because $E(A_j) = E(B_j) = 0$ for all j. For (1), we write

$$E(X_t^2) = \mathrm{Var}(X_t) = \mathrm{Cov}(X_t, X_t)$$

$$= \mathrm{Cov}\left(\sum_j [A_j \cos(v_j t) + B_j \sin(v_j t)], \sum_i [A_i \cos(v_i t) + B_i \sin(v_i t)]\right).$$

Note that for $j \neq i$

$$E(A_j A_i) = 0, \quad E(A_j B_i) = 0, \quad E(B_j A_i) = 0, \quad E(B_j B_i) = 0.$$

Therefore, it must be that

$$\begin{aligned} E(X_t^2) &= \sum_j \mathrm{Cov}(A_j \cos(v_j t) + B_j \sin(v_j t), A_j \cos(v_j t) + B_j \sin(v_j t)) \\ &= \sum_j \left(\cos^2(v_j t) \mathrm{Var}(A_j) + \sin^2(v_j t) \mathrm{Var}(B_j)\right) \\ &= \sum_j \left(\cos^2(v_j t) + \sin^2(v_j t)\right) \sigma_j^2 \\ &= \sum_j \sigma_j^2. \end{aligned}$$

We can proceed in a similar manner for (3):

$$
\begin{aligned}
\operatorname{Cov}(X_s, X_t) &= \operatorname{Cov}\left(\sum_j [A_j \cos(v_j t) + B_j \sin(v_j t)], \sum_i [A_i \cos(v_i s) + B_i \sin(v_i s)]\right) \\
&= \sum_j \operatorname{Cov}(A_j \cos(v_j t) + B_j \sin(v_j t), A_j \cos(v_j s) + B_j \sin(v_j s)) \\
&= \sum_j (\cos(v_j t)\cos(v_j s) \operatorname{Var}(A_j) + \sin(v_j t)\sin(v_j s) \operatorname{Var}(B_j)) \\
&= \sum_j (\cos(v_j t)\cos(v_j s) + \sin(v_j t)\sin(v_j s))\,\sigma_j^2 \\
&= \sum_j \cos[(t-s)v_j]\sigma_j^2
\end{aligned}
$$

which only depends on $|t-s|$. (Recall that cos is an even function.)
(b). Now suppose that $m = 1$ that we may write

$$X_t = A\cos(vt) + B\sin(vt)$$

with

$$E(A) = E(B) = 0, \quad E(AB) = 0, \quad E(A^2) = E(B^2) = \sigma^2.$$

Let $I_t \equiv \{X_s : s \leq t\}$. We show that the process $\{X_t : t \in \mathbb{Z}\}$ is deterministic on $T_1 = \{3, 4, 5, \ldots\}$. To this end, we first note that for any $t \in T_1$, the information set I_{t-1} includes X_1 and X_0. These two observations satisfy

$$\begin{pmatrix} X_1 \\ X_0 \end{pmatrix} = \begin{pmatrix} A\cos(v) + B\sin(v) \\ A \end{pmatrix} = \begin{pmatrix} \cos(v) & \sin(v) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

For simplicity, we treat v as known and assume that $\sin(v) \neq 0$. Then,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \cos(v) & \sin(v) \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_0 \end{pmatrix} = \frac{1}{-\sin(v)} \begin{pmatrix} 0 & -\sin(v) \\ -1 & \cos(v) \end{pmatrix} \begin{pmatrix} X_1 \\ X_0 \end{pmatrix}.$$

It follows that for $t \in T_1$,

$$X_t = \begin{pmatrix} \cos(vt) & \sin(vt) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \cos(vt) & \sin(vt) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \csc(v) & -\cot(v) \end{pmatrix} \begin{pmatrix} X_1 \\ X_0 \end{pmatrix}$$

which can be written as $g_t(I_{t-1})$. $\square$