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TIME SERIES ANALYSIS
EXERCISES
STOCHASTIC PROCESSES 3

1. Let $T \subseteq \mathbb{R}$. Which condition must satisfy a family of joint distribution functions $F(x_1,, \ldots, x_n; t_1,, \ldots, t_n)$ defined for all finite subsets $\{t_1, \ldots, t_n\} \subseteq T$, where $n \geq 1$, in order for these distributions to be the distributions of a stochastic process? [See Brockwell and Davis (1991, Theorem 1.2.1).]

2. Show that any even and positive-definite function $\gamma : \mathbb{Z} \to \mathbb{R}$ is the autocovariance function of a Gaussian stationary process. [See Brockwell and Davis (1991, Theorem 1.5.1).]

3. Consider the function $\gamma : \mathbb{Z} \to \mathbb{R}$ defined by

$$
\begin{aligned}
\gamma(k) &= 1 \ , \quad \text{if } k = 0 \\
&= \rho \ , \quad \text{if } |k| = 1 \\
&= 0 \ , \quad \text{otherwise.}
\end{aligned}
$$

   Show this function is an autocovariance function if and only if $|\rho| \leq 0.5$. [See Brockwell and Davis (1991, Example 1.5.1).]

4. Let $(Z_t : t \in \mathbb{Z})$ be *i.i.d.* $N[0, \ \sigma^2]$ random variables, and let a, b, and c be real constants. Determine which ones of the following processes are second-order stationary. For each stationary process, determine its mean and autocovariance function.

   (a) $X_t = a + b \, Z_t + c \, Z_{t-1}$

   (b) $X_t = a + b \, Z_0$

   (c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$

   (d) $X_t = Z_0 \cos(ct)$

   (e) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

   (f) $X_t = Z_t Z_{t-1}$

5. Let $(Y_t : t \in \mathbb{Z})$ be a second-order stationary process such that $E(Y_t) = 0$, $\forall t$, and let a and b be real constants.

(a) If $X_t = a + bt + s_t + Y_t$, where s_t is a periodic function with period 12, show that the process
$$Z_t = (1 - B)(1 - B^{12})X_t$$
is second-order stationary.

(b) If $X_t = (a + bt)s_t + Y_t$, where s_t is a periodic function with period 12, show that the process
$$Z_t = (1 - B^{12})(1 - B^{12})X_t$$
is second-order stationary.

6. Let $(X_t : t \in \mathbb{Z})$ and $(Y_t : t \in \mathbb{Z})$ be second-order stationary processes uncorrelated with each other, i.e. such that $Cov(X_s, Y_t) = 0, \ \forall s, t$.

(a) Show that the process $X_t + Y_t$ is second-order stationary

(b) Find the autocovariance function of $X_t + Y_t$.

7. Consider the process
$$S_t = \mu + S_{t-1} + u_t , \ t = 1, 2, ...$$
where $S_0 = 0$ and $u_1, u_2, ...$ are $i.i.d.$ random variables with mean zero and variance σ^2.

(a) Determine the mean and covariance function of the process S_t. Is this process strictly stationary? second-order stationary?

(b) Show that the process $Y_t = (1 - B)S_t , t = 1, 2, ...$ is strictly stationary. Compute its mean and autocovariance function.

References

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods.* Springer-Verlag, New York, second edn.