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ECONOMETRICS 1
EXERCISES 1

Covariance matrices

1. Let $\mathbf{X} = (X_1, \ldots, X_k)'$ a $k \times 1$ random vector, α a scalar, $\mathbf{a}$ and $\mathbf{b}$ fixed $k \times 1$ vectors, and A a fixed $g \times k$ matrix. Then, provided the moments considered are finite, show that the following properties hold:
   (a) $\mathsf{E}(\mathbf{X}+\mathbf{a}) = \mathsf{E}(\mathbf{X}) + \mathbf{a}$;
   (b) $\mathsf{E}(\alpha\mathbf{X}) = \alpha\mathsf{E}(\mathbf{X})$;
   (c) $\mathsf{E}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{E}(\mathbf{X})$, $\mathsf{E}(A\mathbf{X}) = A\mathsf{E}(\mathbf{X})$;
   (d) $\mathsf{V}(\mathbf{X}+\mathbf{a}) = \mathsf{V}(\mathbf{X})$;
   (e) $\mathsf{V}(\alpha\mathbf{X}) = \alpha^2\mathsf{V}(\mathbf{X})$;
   (f) $\mathsf{V}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$, $\mathsf{V}(A\mathbf{X}) = A\mathsf{V}(\mathbf{X})A'$;
   (g) $\mathsf{C}(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{b} = \mathbf{b}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$.

2. Let $\mathbf{X} = (X_1, \ldots, X_k)'$ be a random vector with finite second moments and let $\boldsymbol{\Sigma} = \mathsf{V}(\mathbf{X})$ be its covariance matrix. Prove the following properties.
   (a) $\Sigma' = \Sigma$.
   (b) Σ is a positive semidefinite matrix.
   (c) If there exists a nonzero $k \times 1$ fixed vector $\mathbf{a}$ and a constant b such that $\mathbf{X}'\mathbf{a} = b$ with probability one, then $\boldsymbol{\Sigma}\mathbf{a} = 0$ and Σ has rank less than k.
   (d) If the matrix Σ is singular, then there exists a nonzero $k \times 1$ fixed vector $\mathbf{a}$ and a constant b such that $\mathbf{X}'\mathbf{a} = b$ with probability one.