Solution to Econ 763 Midterm (Winter 2017)

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Question 1 [20 points]

(a) *If a random variable has finite second moments, it has finite moments at all highest orders.*
False. A Student's t-distribution with 3 degrees of freedom has a finite second moment but no moments of order 3 or above.

(b) *Any stationary process of order* 5 *is also stationary of order* 2.
True. Suppose that $\{X_t\}_{t \in T}$ satisfies stationarity of order 5. That is,

(i) $E(|X_t|^5) < \infty$, for all $t \in T$;

(ii) $E[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}] = E[X_{t_1+k}^{m_1} \dots X_{t_n+k}^{m_n}]$ for any $k \geq 0$, any subset $\{t_1, \dots, t_n\} \in T^n$ and all non-negative integers $m_1, \dots, m_n$ s.t. $m_1 + m_2 + \dots + m_n \leq 5$.

By Jensen's inequality and the convexity of the map $z \mapsto z^{5/2}$, we have

$$[E(|X_t|^2)]^{5/2} \leq E\left[\left(|X_t|^2\right)^{5/2}\right] = E\left(|X_t|^5\right) \overset{\text{(i)}}{<} \infty$$

implying that $E(|X_t|^2) < \infty$. Next, for any $k \geq 0$, any subset $\{t_1, \dots, t_n\} \in T^n$ and all non-negative integers $m_1, \dots, m_n$ s.t. $m_1 + m_2 + \dots + m_n \leq \mathbf{2}$, it is trivially true that $m_1 + m_2 + \dots + m_n \leq \mathbf{5}$. Therefore, by (ii) above, we have $E[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}] = E[X_{t_1+k}^{m_1} \dots X_{t_n+k}^{m_n}]$.

(c) *Any strictly stationary process is in* L_2.
False. Suppose that $\{y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. random variables each of which follows a Student's t-distribution with 2 degrees of freedom. The i.i.d. assumption implies strict stationarity. Yet, each y_t does not possess a finite second moment.

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(d) *The Wold theorem holds for finite-order moving average processes but not autoregressive processes.*
False. The theorem holds for all second-order stationary processes which do include moving average processes and which do **not** exclude autoregressive processes.

(e) *Non-invertible moving processes have no covariance generating function.*
False. For example, take $X_t = u_t - 2u_{t-1}$ where $u_t \sim \mathrm{WN}(0, \sigma^2)$. Then, $\{X_t\}$ is not invertible but has covariance generating function $z \mapsto \sigma^2(1-2z)(1-2z^{-1})$.

Question 2 [20 points]

Let $\gamma(k)$ the autocovariance function of second-order stationary process on the integers. Prove that:

(a) $\gamma(0) = \mathrm{Var}(X_t)$ and $\gamma(k) = \gamma(-k)$, for all $k \in \mathbb{Z}$;

(b) $|\gamma(k)| \leq \gamma(0)$, $\forall k \in \mathbb{Z}$;

(c) the function $\gamma(k)$ is positive semi-definite.

Proof.

(a) By definition, we have $\mathrm{Cov}(X_s, X_t) = \gamma(t-s)$ for all $s, t \in T$. By the second-order stationarity, the autocovariance function γ is well-defined. In particular,

$$\gamma(0) = \mathrm{Cov}(X_t, X_t) = \mathrm{Var}(X_t)$$

and (s below is any integer)

$$\gamma(k) = \mathrm{Cov}(X_s, X_{s+k}) = \mathrm{Cov}(X_{s+k}, X_s) = \gamma(s - (s+k)) = \gamma(-k).$$

(b) That $|\gamma(k)| \leq \gamma(0)$ is a consequence of the Cauchy-Schwarz inequality. We provide here a direct proof for completeness. With a fixed $k \in \mathbb{Z}$ and any $z \in \mathbb{R}$, we have

$$0 \leq \mathrm{Var}(X_s - zX_{s+k}) = \mathrm{Var}(X_s) - 2z\,\mathrm{Cov}(X_s, X_{s+k}) + z^2\,\mathrm{Var}(X_{s+k}).$$

The rightmost expression above is a quadratic polynomial in z so it is nonnegative for all real z iff the discriminant is nonpositive:

$$0 \leq [-2\,\mathrm{Cov}(X_s, X_{s+k})]^2 - 4\,\mathrm{Var}(X_s)\,\mathrm{Var}(X_{s+k}) \iff \gamma(k)^2 \leq \gamma(0)^2$$

from which $|\gamma(k)| \leq |\gamma(0)| = \gamma(0)$ follows immediately.

(c) We have to show that: for any positive integer N and for all vectors $a = (a_1, \ldots, a_N)' \in \mathbb{R}^N$ and $\tau = (t_1, \ldots, t_N)' \in T^N$, it holds that $\sum_{i=1}^N \sum_{j=1}^N a_i a_j \gamma(t_i - t_j) \geq 0$. This follows by considering $Z \equiv a_1 X_{t_1} + \cdots a_N X_{t_N}$. We have

$$0 \leq \text{Var}(Z) = \text{Cov}(Z, Z) = \text{Cov}\left(\sum_{i=1}^N a_i X_{t_i}, \sum_{j=1}^N a_j X_{t_j}\right)$$
$$= \sum_{i=1}^N \sum_{j=1}^N \text{Cov}(a_i X_{t_i}, a_j X_{t_j}) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \,\text{Cov}(X_{t_i}, X_{t_j}) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \gamma(t_i - t_j).$$

□

Question 3 [60 points]

Consider the following models:

$$X_t = 10 + u_t - 0.75 u_{t-1} + 0.125 u_{t-2}, \tag{1}$$

where $\{u_t : t \in \mathbb{Z}\}$ is an i.i.d. $N(0, 1)$ sequence. Answer the following questions.

(a) Is this model stationary? Why?

Answer. Stationarity is automatic for all finite-order MA processes. □

(b) In this model invertible? Why?

Answer. Write the process in (1) as

$$X_t = \mu + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, \quad \mu = 10, \quad \theta_1 = \frac{3}{4}, \quad \theta_2 = -\frac{1}{8}.$$

This MA(2) process is invertible because $\theta(z) = 1 - \theta_1 z - \theta_2 z^2$ has 2 roots: 2 and 4, which are both outside the unit circle. □

(c) Compute:

i. $E(X_t)$;

ii. $\gamma(k)$, $k = 1, \ldots, 8$;

iii. $\rho(k)$, $k = 1, \ldots, 8$.

Answer. We have:

i. $E(X_t) = \mu = 10$.

ii. We have

$$\gamma(0) = \text{Var}(X_t) = \sigma^2(1+\theta_1^2+\theta_2^2) = \frac{101}{64}$$
$$\gamma(1) = \sigma^2(-\theta_1+\theta_1\theta_2) = \frac{-27}{32},$$
$$\gamma(2) = \sigma^2(-\theta_2) = \frac{1}{8},$$
$$\gamma(3) = \gamma(4) = \cdots = \gamma(8) = 0.$$

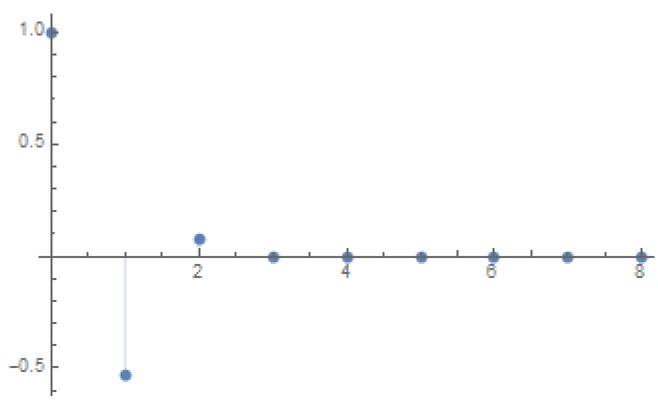
iii. It follows that

$$\rho(0) = 1;$$
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = -\frac{54}{101},$$
$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{8}{101},$$
$$\rho(3) = \rho(4) = \cdots = \rho(8) = 0.$$

□

(d) Graph $\rho(k)$, $k = 1, \ldots, 8$.

Answer.



□

(e) Find the coefficients of u_t, u_{t-1}, u_{t-2}, u_{t-3}, and u_{t-4} in the moving average representation of X_t.

Answer. This is trivial for an MA(2) process: the coefficients of u_t, u_{t-1}, u_{t-2}, u_{t-3}, and u_{t-4}, respectively, are 1, $-\frac{3}{4}$, $\frac{1}{8}$, 0, and 0. □

(f) Find the autocovariance generating function of X_t.

Answer. The autocovariance generating function is

$$\gamma_x(z) = 1^2\theta(z)\theta(1/z) = \frac{(8-6z+z^2)(1-6z+8z^2)}{64z^2}.$$

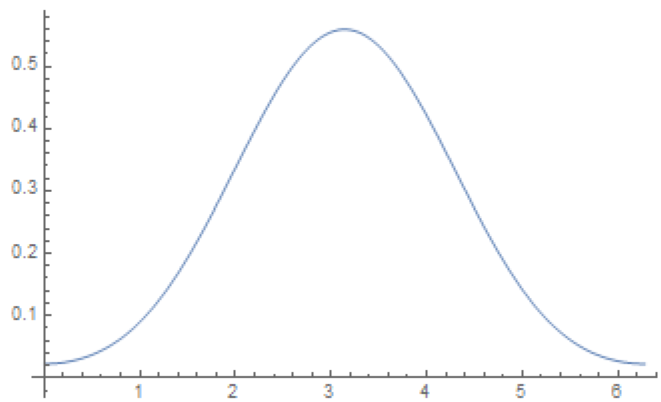
□

(g) Find and graph the spectral density of X_t.

Answer. The spectral density is

$$f_x(\omega) = \frac{1^2}{2\pi}\theta(e^{i\omega})\theta(e^{-i\omega}) = \frac{101-108\cos(\omega)+16\cos(2\omega)}{128\pi}.$$

which we can graph as



□

(h) Compute the first two partial autocorrelations of X_t.

Answer. Let $\phi(k)$ denote the partial autocorrelation of order k. Then,

$$\phi(1) = 1^{-1}\rho(1) = -\frac{54}{101},$$
$$\begin{pmatrix} \cdots \\ \phi(2) \end{pmatrix} = \begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix} = \begin{pmatrix} \cdots \\ -\frac{68}{235} \end{pmatrix}.$$

□

(i) If $X_{10} = 1$ and assuming the parameters of the model are known, can you compute the best linear forecasts of X_{10}, X_{11}, X_{12}, and X_{13} based on X_{10} (only)? If so, compute these.

Answer. The best linear forecast of X_{10} is trivially 1 itself. The best linear forecast for X_{11} based on X_{10} is

$$\begin{aligned} P_L(X_{11}|X_{10}) &= (E(X_{11}) - E(X_{10})\rho(1)) + \rho(1)X_{10} \\ &= \left(10 + 10\frac{54}{101}\right) - \frac{54}{101}1 = \frac{1496}{101} = 14.8119. \end{aligned}$$

Similarly,

$$\begin{aligned} P_L(X_{12}|X_{10}) &= (E(X_{12}) - E(X_{10})\rho(2)) + \rho(2)X_{10} \\ &= \left(10 - 10\frac{8}{101}\right) + \frac{8}{101}1 = \frac{938}{101} = 9.28713, \\ P_L(X_{13}|X_{10}) &= (E(X_{13}) - E(X_{10})\rho(3)) + \rho(3)X_{10} = (10 - 10 \times 0) + 0 \times 1 = 10. \end{aligned}$$

□

(j) If $X_{10} = 1$, $u_{10} = 2$, $u_9 = 1$, $u_8 = 0.99$, $u_7 = 1.2$, and assuming the parameters of the model are known, can you compute the best linear forecasts of X_{11}, X_{12}, X_{13} based on the history of the process up to X_{10}? If so, compute these.

Answer. Let Ω_t be the information set that is available at period 10. Then, because

$$\begin{aligned} X_{11} &= 10 + u_{11} - 0.75u_{10} + 0.125u_9, \\ X_{12} &= 10 + u_{12} - 0.75u_{11} + 0.125u_{10}, \\ X_{13} &= 10 + u_{13} - 0.75u_{12} + 0.125u_{11}; \end{aligned}$$

we have

$$\begin{aligned} P_{10}X_{11} &= 10 + E(u_{11}|\Omega_{10}) - 0.75E(u_{10}|\Omega_{10}) + 0.125E(u_9|\Omega_{10}) \\ &= 10 + 0 - 0.75 \times u_{10} + 0.125 \times u_9 = 8.625, \\ P_{10}X_{12} &= 10 + 0 - 0.75 \times 0 + 0.125 \times u_{10} = 10.25, \\ P_{10}X_{13} &= 10 + 0 - 0.75 \times 0 + 0.125 \times 0 = 10. \end{aligned}$$

□

References