Solution to Econ 763 Assignment 2 (Winter 2017)  
Instructor: Jean-Marie Dufour*

Vinh Nguyen†

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Problem 1 (20 points)

Grading remarks: 5 points each for (a)–(d)

Suppose we have the formal series

\[ \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \]

where \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \). For a fixed \( t \), we can in general write

\[ \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = Y_j + \sum_{j=1}^{\infty} Y_j \]

where \( Y_j \equiv \psi_j u_{t-j} \). In particular, the dependence of \( Y_j \) on \( t \) has been suppressed in the notation. Note that \( E(Y_j) = 0 \), \( E(Y_j^2) = \psi_j^2 \sigma^2 < \infty \) so that \( Y_j \in L_2 \), and \( E(Y_i Y_j) = 0 \) for \( i \neq j \).

(a) Convergence in mean of order 2

Proposition 4.2.6 in Dufour (2008b) implies that if

\[ \infty > \sum_{j=-\infty}^{\infty} (E[Y_j^2])^{1/2} = \sum_{j=-\infty}^{\infty} (\psi_j^2 E(u_{t-j}^2))^{1/2} = \sigma \sum_{j=-\infty}^{\infty} |\psi_j| \]  \hspace{1cm} (1)

then there exists random variables \( Y^- \) and \( Y^+ \) such that

\[ \sum_{j=-m}^{0} Y_j \xrightarrow{m \to \infty} Y^- \]
\[ \sum_{j=0}^{n} Y_j \xrightarrow{n \to \infty} Y^+ \]
We can thus write \( Y^- = \sum_{j=-\infty}^{0} Y_j \) and \( Y^+ = \sum_{j=1}^{\infty} Y_j \). Moreover, we have
\[
\sum_{j=-m}^{0} Y_j + \sum_{j=0}^{n} Y_j \xrightarrow{m,n \to \infty} Y^- + Y^+ \equiv Y.
\]

Having shown convergence, we are now justified in writing
\[
Y = \sum_{j=-\infty}^{\infty} Y_j = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}.
\]

**Remark**: what the above has shown is that \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \) is sufficient for the convergence in mean of order 2 of \( \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \). A different result from Dufour (2008b) (Proposition 4.3.1) gives another sufficient condition
\[
\infty > \sum_{j=-\infty}^{\infty} E[|Y_t|^2] = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j^2 \iff \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty.
\]  

We note that (1) is a strictly stronger condition than (2): the former implies the latter but the reverse implication fails. To see that, the convergence of \( \sum_{j=-\infty}^{\infty} |\psi_j| \) implies that there is \( N \) sufficiently large that for \( |n| \geq N \), we have \( |\psi_j| < 1 \). Then, for \( n, m > N \), we have
\[
\sum_{j=-m}^{n} \psi_j^2 = \sum_{j=-N}^{N} \psi_j^2 + \sum_{j=N+1}^{n} \psi_j^2 + \sum_{j=-m}^{-N-1} \psi_j^2 \\
\leq \sum_{j=-N}^{N} \psi_j^2 + \sum_{j=N+1}^{n} \psi_j^2 + \sum_{j=-m}^{-N-1} |\psi_j|.
\]

When we let \( m, n \to \infty \), absolute summability (i.e. (1)) implies that the second line above converges, which in turn gives the convergence of \( \sum_{j=-\infty}^{\infty} \psi_j^2 \). To see that square-summability doesn’t imply absolute summability, consider
\[
\psi_j = 0 \quad \forall j \leq 0, \quad \psi_j = \frac{1}{j} \quad \forall j \geq 1.
\]

Then
\[
\sum_{j=-\infty}^{\infty} \psi_j^2 = \sum_{j=-\infty}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \sum_{j=-\infty}^{\infty} \psi_j = \sum_{j=1}^{\infty} \frac{1}{j} = +\infty.
\]

**(b) Convergence in mean of order \( r \)**

Following the same approach above and Proposition 4.2.6 (Dufour (2008b)), we may infer that for \( r \geq 1 \), the condition
\[
\infty > \sum_{j=-\infty}^{\infty} (E[|\psi_j u_{t-j}|^r])^{1/r} = E(|u_t|^r)^{1/r} \sum_{j=-\infty}^{\infty} |\psi_j| \iff \sum_{j=-\infty}^{\infty} |\psi_j| < \infty
\]  

(3)
is sufficient for \( \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \) to converge in mean of order \( r \). Of course, here we also need each \( u_t \) to be in \( L_r \).

For \( r < 1 \), we also appeal to Proposition 4.2.6. To be specific, that proposition tells us that for \( \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \) to converge in mean (i.e. in \( L_1 \)), it also suffices to have \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \) (and that for each \( t \), \( E(|u_t|) \) is finite but this follows because \( E(u_t^2) \) is finite.) But convergence in \( L_1 \) implies convergence in \( L_r \) for \( r < 1 \), so the same sufficient condition is enough for \( \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \) to converge in mean of order \( r < 1 \).

(c) Almost sure convergence

Proposition 4.2.6 again gives us a sufficient condition

\[
\sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{4}
\]

Proposition 4.3.1 competes to give another sufficient condition

\[
\sum_{j=1}^{\infty} (\log j)^2 \psi_j^2 < \infty, \quad \sum_{j=-\infty}^{-1} (\log(-j))^2 \psi_j^2 < \infty \tag{5}
\]

We see that (5) does not imply (4). For example, when \( \psi_j = 0 \) for \( j \leq 0 \) and \( \psi_j = \frac{1}{j} \) for \( j \geq 1 \), we have

\[
\sum_{j=1}^{\infty} (\log j)^2 \psi_j^2 = \sum_{j=1}^{\infty} \frac{(\log j)^2}{j^2} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |\psi_j| = \sum_{j=1}^{\infty} \frac{1}{j} \to \infty.
\]

(d) Convergence in probability

Since convergence in probability is implied by convergence in mean of order \( r \) (\( r > 0 \)) and almost sure convergence, each of the conditions (2), (4), and (5) will be sufficient here.

Problem 2 (10 points)

Grading remarks: 5 points each for (a) and (b)

Consider an \( MA(1) \) model

\[
X_t = \bar{\mu} + u_t - \theta u_{t-1}, \quad t \in \mathbb{Z}
\]

where \( u_t \sim WN(0, \sigma^2) \) and \( \sigma^2 > 0 \).

(a) The first autocorrelation of this model cannot be greater than 0.5 in absolute value.
Proof. We have
\[
\text{Cov}(X_t, X_{t+1}) = E[(u_t - \theta u_{t-1})(u_{t+1} - \theta u_t)] = -\theta E(u_t^2) = -\theta \sigma^2,
\]
\[
\text{Var}(X_t) = \text{Var}(u_t) + \theta^2 \text{Var}(u_{t-1}) = (1 + \theta^2)\sigma^2.
\]
This implies
\[
|\rho(1)| = \left| \frac{\text{Cov}(X_t, X_{t+1})}{\text{Var}(X_t)} \right| = \frac{|\theta|}{1 + \theta^2}
\]
which is less than or equal to \( \frac{1}{2} \) because
\[
\frac{|\theta|}{1 + \theta^2} \leq \frac{1}{2} \iff 2|\theta| \leq 1 + \theta^2 \iff (|\theta| - 1)^2 \geq 0.
\]
\( \square \)

(b) Values of the model parameters for which this upper bound is attained.

Answer. As shown in (a), we have
\[
2(1 + \theta^2) \left( \frac{1}{2} - |\rho(1)| \right) = (1 + \theta^2)(|\theta| - 1)^2 \geq 0
\]
which equals 0 iff \( |\theta| = 1 \). That is, when \( \theta = \pm 1 \), the absolute value of the first autocorrelation equals \( \frac{1}{2} \).
\( \square \)

Problem 3 (72 points)
Grading remarks: for each process, 3 points each for (a)–(f) and \( 4 \times 18 = 72 \) points total
Let \( \{X_t : t \in \mathbb{Z}\} \) be an \( MA(q) \) process. For \( q = 3, 4, 5, 6 \), check whether the following inequalities are correct:

(a) \( |\rho(1)| \leq 0.75; \)
(b) \( |\rho(2)| \leq 0.90; \)
(c) \( |\rho(3)| \leq 0.90; \)
(d) \( |\rho(4)| \leq 0.90; \)
(e) \( |\rho(5)| \leq 0.90; \)
(f) \( |\rho(6)| \leq 0.90. \)
A general $MA(q)$ process can be written as

$$X_t = \mu + u_t + \sum_{t=1}^{q} \theta_j u_{t-j} = \mu + \theta(L) u_t \quad \text{with} \quad \theta(L) = 1 + \theta_1 L - \ldots - \theta_q L^q.$$  

From the lecture notes (Dufour (2008a)), the autocorrelation coefficients can be computed as follows

$$\rho(k) = \frac{\left(\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k}\right)}{1 + \sum_{j=1}^{q} \theta_j^2}, \quad 1 \leq k \leq q$$

$$= 0, \quad k \geq q + 1.$$  

In particular, the autocorrelations vanish for $k \geq q + 1$. Moreover, formula (6.12) from the lecture notes gives us

$$|\rho(k)| \leq B(q,k) \equiv \cos \left(\frac{\pi \lfloor q/k \rfloor + 2}{q/k}\right).$$

Plotting $B(q,k)$ for various $q$ and $k$ gives

![Figure 1: Upperbounds for autocorrelations of some MA processes](image)

(a) $MA(3)$     (b) $MA(4)$

(c) $MA(5)$     (d) $MA(6)$
From the Figure 1, we know that (b)–(f) must hold, but let’s verify this algebraically. Because \( \rho(k) = 0 \) for \( k \geq 4 \), the inequalities in (d)–(f) hold automatically. For (a)–(c), we write
\[
\rho(1) = \frac{\theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2},
\]
\[
\rho(2) = \frac{\theta_2 + \theta_1 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2},
\]
\[
\rho(3) = \frac{\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}.
\]

From these, (c) holds because
\[
2|\theta_3| \leq 1 + \theta_3^2 \leq 1 + \theta_1^2 + \theta_2^2 + \theta_3^2 \implies |\rho(3)| \leq \frac{1}{2} < 0.90.
\]

In a similar manner, we can use the inequality \( 2ab \leq a^2 + b^2 \) to infer
\[
2|\theta_2 + \theta_1 \theta_3| \leq 2|\theta_2| + |\theta_1||\theta_3| \leq 1 + \theta_2^2 + \theta_1^2 + \theta_3^2 \implies |\rho(2)| \leq \frac{1}{2} < 0.90.
\]

So (b) indeed holds. As the figure suggests however, (a) can fail. And it does when we set \( \theta_1 = \theta_2 = \theta_3 = \theta = \frac{3}{2} \) so that
\[
|\rho(1)| = \frac{\theta(1 + 2\theta)}{1 + 3\theta^2} = \frac{24}{31} > \frac{24}{32} = 0.75.
\]

Again, Figure 1 says that (b)–(f) are true whereas (a) may fail. For (e)–(f), the implications are immediate because \( \rho(5) = \rho(6) = 0 \). For the rest, we write
\[
\rho(1) = \frac{\theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3 + \theta_3 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},
\]
\[
\rho(2) = \frac{\theta_2 + \theta_1 \theta_3 + \theta_2 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},
\]
\[
\rho(3) = \frac{\theta_3 + \theta_1 \theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},
\]
\[
\rho(4) = \frac{\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}.
\]

Because \( |\theta_4| \leq \frac{1}{2}(1 + \theta_2^2) \), it’s obvious that \( |\rho(4)| \leq \frac{1}{2} < 0.90 \). Similarly, \( |\theta_3| \leq \frac{1}{2}(1 + \theta_3^2) \) and \( |\theta_1||\theta_4| \leq \frac{1}{2}(\theta_1^2 + \theta_3^2) \) imply that \( |\rho(3)| \leq \frac{1}{2} < 0.90 \). To prove \( |\rho(2)| \leq 0.90 \) we can assume WLOG that \( \theta_i \geq 0 \) so that \( |\rho(2)| \leq 0.90 \) is equivalent to
\[
9 + 9\theta_1^2 + 9\theta_2^2 + 9\theta_3^2 + 9\theta_4^2 \geq 10\theta_2 + 10\theta_1\theta_3 + 10\theta_2\theta_4.
\]
Noting that \(9\theta_1^2 + 9\theta_3^2 - 10\theta_1\theta_3 = (2\theta_1)^2 + (2\theta_2)^2 + 5(\theta_1 - \theta_3)^2\), we only need to prove
\[
9 + 9\theta_2^2 + 9\theta_4^2 \geq 10\theta_2 + 10\theta_2\theta_4.
\]

We can treat \((*)\) as an inequality for \(\theta_4\) equals some fixed \(y \geq 0\) and while \(\theta_2 = x \geq 0\) is allowed to vary. That is, \((*)\) follows if we can show that
\[
9 + 9x^2 + 9y^2 \geq 10x + 10xy = (10 + 10y)x
\]
for all \(x, y \geq 0\). With \(y \geq 0\) fixed, the LHS above is convex in \(x\) whereas the RHS is linear. The derivative (w.r.t. to \(x\)) of the RHS is \(10 + 10y\) whereas the derivative of the LHS is \(18x\) which equals \(9 + 10y\) when \(x = \frac{5}{9}(1 + y)\). At this value of \(x\), \((**\)) is equivalent to
\[
81 + 81y^2 + 25(y + 1)^2 \geq 50(y + 1)^2 \iff 0 \leq 56y^2 - 50y + 56
\]
which is true because \(56y^2 - 50y + 56 = 31y^2 + 31 + 25(y - 1)^2\). Due to the convexity observation from the previous paragraph, that the inequality holds for \(x = \frac{5}{9}(1 + y)\) is enough for \((**)\) to hold for all \(x \geq 0\) given \(y\) is fixed. As \(y\) is arbitrary, \((**\)) and, thus, \((*)\) must hold generally. In other words, \(|\rho(2)| \leq \frac{9}{10}\) is true.

Finally, (a) fails when we set \(\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta\) so that
\[
|\rho(1)| = \frac{\theta + 3\theta^2}{1 + 4\theta^2} = \frac{232}{281} > \frac{210}{280} = 0.75.
\]

\(MA(5)\)

Figure 1 says that (b)–(f) are true while (a) may fail. \(\rho(6) = 0\) so (f) is immediate. For (a)–(e), we write
\[
\rho(1) = \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4 + \theta_4\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},
\]
\[
\rho(2) = \frac{\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \theta_3\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},
\]
\[
\rho(3) = \frac{\theta_3 + \theta_1\theta_4 + \theta_2\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},
\]
\[
\rho(4) = \frac{\theta_4 + \theta_1\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},
\]
\[
\rho(5) = \frac{\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2}.
\]

Using the same techniques as above, we can again show that \(|\rho(3)|, |\rho(4)|, \text{ and } |\rho(5)|\) are less than or equal to \(\frac{5}{7} < 0.90\). The autocorrelation of order 1 \(\rho(1)\) can exceed 0.75 in absolute value: when \(\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta = \frac{3}{2}\), we have
\[
|\rho(1)| = \frac{\theta + 4\theta^2}{1 + 5\theta^2} = \frac{6}{7} > 0.75.
\]
As with the \( MA(4) \) case, \( \rho(2) \) poses a more challenging problem. The algebra seems intimidating so we settle with formula (6.12) from Dufour (2008a):

\[
|\rho(2)| \leq \cos \left( \frac{\pi}{[q/k] + 2} \right) \bigg|_{q=5, k=2} = \frac{1}{\sqrt{2}} < 0.9.
\]

\( MA(6) \)

We get no free lunch with this one as none of the correlation is 0. The standard formulas give

\[
\begin{align*}
\rho(1) &= \frac{\theta_1 + \theta_2 + \theta_3 \theta_4 + \theta_5 \theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(2) &= \frac{\theta_2 + \theta_3 \theta_4 + \theta_5 \theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(3) &= \frac{\theta_3 + \theta_4 \theta_5 + \theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(4) &= \frac{\theta_4 + \theta_5 \theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(5) &= \frac{\theta_5 + \theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(6) &= \frac{\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}.
\end{align*}
\]

A quick glance gives \( |\rho(4)|, |\rho(5)| \) and \( |\rho(6)| \) are no more than \( \frac{1}{2} < 0.9 \). To find the counter example for \( \rho(1) \), we set \( \theta_1 = \ldots = \theta_6 = \theta = \frac{\pi}{5} \) to obtain

\[
|\rho(1)| = \frac{\theta + 5\theta^2}{1 + 6\theta^2} = \frac{360}{409} > \frac{360}{480} = 0.75.
\]

The algebra for \( \rho(2) \) and \( \rho(3) \) looks scary so we again use (6.12) from Dufour (2008a):

\[
\begin{align*}
|\rho(2)| &\leq \cos \left( \frac{\pi}{[q/k] + 2} \right) \bigg|_{q=6, k=2} = \frac{1}{4} (1 + \sqrt{5}) < 0.9, \\
|\rho(3)| &\leq \cos \left( \frac{\pi}{[q/k] + 2} \right) \bigg|_{q=6, k=3} = \frac{1}{\sqrt{2}} < 0.9.
\end{align*}
\]

**Problem 4 (300 points)**

Grading remarks: for each process, 2 points for (a), 2 points for (b), 7 \((1 + 4 + 2)\) points for (c), 3 points for (d), 5 points for (e), 2 points for (f), 5 \((2 + 3)\) points for (g), 4 points for (h), and so \( 6 \times 30 = 180 \) points total
Some general results for ARMA\((p, q)\) \((p, q \text{ finite})\)

For some finite and positive integers \(p\) and \(q\), we consider a process \(\{X_t : t \in \mathbb{Z}\}\) which satisfies the equation

\[
X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}
\]

(6)

where \(\{u_t : t \in \mathbb{Z}\}\) is a homoskedastic white noise with common variance \(\sigma^2\). Using operational notation, we can define \(\varphi(B) = 1 - \sum_{j=1}^{p} \varphi_j B^j\) and \(\theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j\) and write

\[
\varphi(B) X_t = \bar{u} + \theta(B) u_t.
\]

(7)

(1) **Stationarity condition:** if the polynomial \(\varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p\) has all its roots outside the unit circle, the equation (6) has one and only one weakly stationary solution, which can be written

\[
X_t = \mu + [\varphi(B)]^{-1} \theta(B) u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}
\]

(8)

where

\[
\mu = \frac{\bar{\mu}}{\varphi(B)} = \frac{\bar{\mu}}{1 - \sum_{j=1}^{p} \varphi_j},
\]

\[
\frac{\theta(B)}{\varphi(B)} \equiv \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j.
\]

(2) The \(\psi_j\) coefficients are obtained by solving the equation \(\varphi(B) \psi(b) = \theta(B)\):

\[
\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right) \left(\sum_{j=0}^{\infty} \psi_j B^j\right) = 1 - \sum_{j=1}^{q} \theta_j B^j
\]

(9)

and comparing powers of \(B\)'s on both sides. For examples, (below we define \(\theta_0 = -1\))

\[
\psi_0 = -\theta_0 = 1,
\]

\[
\psi_1 - \varphi_1 = -\theta_1,
\]

\[
\psi_2 - \varphi_1 \psi_1 - \varphi_2 = -\theta_2,
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots
\]

\[
\psi_j - \sum_{k=1}^{j} \varphi_k \psi_{j-k} = -\theta_j, \quad (j = 0, 1, \ldots, q)
\]

If we define \(\psi_j = 0\) for \(j < 0\) then the last line above can be rewritten as \(\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} = -\theta_j\) for \(j = 0, \ldots, q\). For \(j > q\), things get slightly trickier. The
advantage of this re-expression is that for \( j > q \), we can also write 
\[
\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} = 0.
\]

Thus, a convenient algorithm for solving for \( \psi_j \) is that:

(i) define \( \psi_{-p} = \psi_{-(p-1)} = \ldots = \psi_1 = 0 \),
(ii) for \( j = 0, 1, \ldots, q \), recursively compute 
\[
\psi_j = -\theta_j + \sum_{k=1}^{p} \varphi_k \psi_{j-k},
\]
(iii) for \( j > q \), continue the recursion
\[
\psi_j = \sum_{k=1}^{p} \varphi_k \psi_{j-k}.
\]

(c) **Invertibility:** If the ARMA process (7) is second-order stationary, then the process \( \{X_t\} \) satisfies an equation of the form
\[
\sum_{j=0}^{\infty} \tilde{\phi}_j X_{t-j} = \tilde{\mu} + u_t
\]
iff the roots of the polynomial \( \theta(B) \) are outside the unit circle. Further, when the representation above exists, we have
\[
\tilde{\phi}(B) = \theta(B)^{-1} \varphi(B), \quad \tilde{\mu} = \theta(B)^{-1} \bar{\mu} = \frac{\bar{\mu}}{1 - \sum_{j=1}^{q} \theta_j}.
\]
In particular, any stationary AR\((p)\) process is invertible. Note that invertibility is actually a separate concept from stationarity. In Box et al. (2008), a linear process \( X_t = \mu + \sum_{j=1}^{\infty} \psi_j u_{t-j} \) is invertible if \( \sum_{j=0}^{\infty} |\pi_j| < \infty \), where \( \pi(B) = \psi^{-1}(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j \).

(d) **Autocovariances and autocorrelations:** Suppose that

(i) the polynomial \( \varphi(z) \) has roots outside the unit circle and the process \( X_t \) the unique stationary solution to \( \varphi(B)X_t = \bar{\mu} + \theta(B)u_t \),
(ii) \( E(X_{t-j}u_t) = 0 \) for all \( j \geq 1 \).

By the stationarity assumption, \( E(X_t) = \mu \) for some \( \mu \) and for all \( t \). This \( \mu \) satisfies
\[
\mu = E(X_t), \forall t \implies \varphi(B)\mu = E[\varphi(B)X_t] = \bar{\mu} \implies \mu = \frac{\bar{\mu}}{1 - \sum_{j=1}^{p} \varphi_j}.
\]

Now, let us define \( Y_t = X_t - \mu \) so that \( E(Y_t) = 0 \) and \( \varphi(B)Y_t = \theta(B)u_t \). It follows that for \( k > 0 \)
\[
Y_{t+k} = \sum_{j=1}^{p} \varphi_j Y_{t+k-j} + u_{t+k} - \sum_{j=1}^{q} \theta_j u_{t+k-j},
\]
\[
\implies E[Y_t Y_{t+k}] = \sum_{j=1}^{p} \varphi_j E[Y_t Y_{t+k-j}] + E[Y_t u_{t+k}] - \sum_{j=1}^{q} \theta_j E[Y_t u_{t+k-j}],
\]
which implies

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) - \sum_{j=1}^{q} \theta_j \gamma_{xu}(k-j)$$

(10)

where

$$\gamma_{xu}(k) = E(Y_t u_{t+k}) = \begin{cases} 0 & \text{if } k \geq 1 \\ \sigma^2 & \text{if } k = 0 \end{cases}$$

and $\gamma_{xu}(k) \neq 0$ in general for $k \leq 0$. That is, for $1 \leq k \leq q$,

$$\gamma_{xu}(-k) = E(Y_t u_{t-k})$$

$$= E \left[ \left( \sum_{j=1}^{p} Y_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \right) u_{t-k} \right]$$

$$= \sum_{j=1}^{p} \gamma_{xu}(-k+j) - \theta_k \sigma^2.$$ 

As $j$ in the last line above is strictly positive, $-k+j > -k$ so that $\gamma_{xu}$ can be computed backwards recursively. Once we have found $\gamma_{xu}$, we can solve (10) and

$$\gamma(0) = \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)$$

for $\gamma(0), \gamma(1), \ldots, \gamma(p)$ in terms of the ARMA coefficients. Then for $k > p$, $\gamma(k)$ can be computed using (10). Finally, the autocorrelation $\rho(0)$ is simply $\frac{\gamma(k)}{\gamma(0)}$.

(e) **Partial autocorrelations**: the partial autocorrelation of order $k$, denoted by $\phi(k)$, is computed as follows: first, we define

$$\Phi(k) \equiv \begin{pmatrix} 1 & \rho(1) & \ldots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & \ldots & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(k-2) & \rho(k-3) & \ldots & 1 & \rho(1) \\ \rho(k-1) & \rho(k-2) & \ldots & \rho(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k-1) \\ \rho(k) \end{pmatrix}.$$ 

(11)

Then, $\phi(k)$ is just the $k$-th entry of $\Phi(k)$.

The **AR(1) process** $X_t = 0.5 X_{t-1} + u_t$

Write this as $(1 - \varphi_1)X_t = \bar{u} + u_t$ where $\varphi_1 = 0.5$ and $\bar{u} = 0$. Here, $u_t \sim N(0, \sigma^2)$ where $\sigma = 1$. 

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(a) The process is stationary because $\varphi(z) = 1 - 0.5z$ has root $z = 2$ which is outside the unit circle.

(b) The process is invertible, as is any other AR$(p)$ process for some finite $p$.

(c) (i) $E(X_t) = \frac{\varphi_1}{1-\varphi_1} = \frac{0}{1-0.5} = 0$,
(ii) Using formulae (7.47) and (7.46) from Dufour (2008a), we have
\[
\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1 - \varphi_1^2} = \frac{1}{0.75} = \frac{4}{3};
\]
\[
\gamma(k) = \varphi_1^k \gamma(0) = \frac{4}{2^k 3^k}.
\]
(iii) The autocorrelations are
\[
\rho(0) = 1, \quad \rho(k) = \varphi_1^k = \frac{1}{2^k}.
\]

(d) We can plot $\rho(k)$ for $k = 0, \ldots, 8$:

(e) Write MA$(\infty)$ representation as $X_t = \psi(B)u_t$ where $\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j$. Because
\[
\frac{1}{1 - \varphi_1 B} = 1 + \varphi_1 B + \varphi_1^2 B^2 + \varphi_1^3 B^3 + \varphi_1^4 B^4 + \cdots,
\]
we have
\[
\psi_0 = 1, \\
\psi_1 = \varphi_1 = 0.5; \\
\psi_2 = \varphi_1^2 = 0.25; \\
\psi_3 = \varphi_1^3 = 0.125; \\
\psi_4 = \varphi_1^4 = 0.0625.
\]
(f) With \( \psi(z) = \frac{1}{1-\varphi_1 z} \) as defined above, then
\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}) \\
= \frac{\sigma^2}{(1 - \varphi_1 z)(1 - \varphi_1 z^{-1})} \\
= \frac{1}{(1 - 0.5z)(1 - 0.5/z)} \\
= \frac{4z}{(2 - z)(2z - 1)}.
\]

(g) By Proposition 11.14 from Dufour (2008a), we have
\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(\exp(i\omega)) \psi(\exp(-i\omega)) \\
= \frac{\sigma^2}{2\pi (1 - \varphi_1 \exp(i\omega))(1 - \varphi_1 \exp(-i\omega))} \\
= \frac{1}{2\pi [1 - 0.5 \exp(i\omega)][1 - 0.5 \exp(-i\omega)]} \\
= \frac{2}{\pi (5 - 4 \cos(\omega))}.
\]
Plotting it yields:


(h) Using the formula (11) four times, we get
\[
\phi(1) = \frac{1}{2}, \quad \phi(2) = \phi(3) = \phi(4) = 0.
\]
The AR(1) process $X_t = 10 - 0.75X_{t-1} + u_t$

Write this as $(1 - \varphi_1)X_t = \bar{u} + u_t$ where $\varphi_1 = -0.75$ and $\bar{u} = 10$. Here, $u_t \sim N(0, \sigma^2)$ where $\sigma = 1$.

(a) The process is stationary because $\varphi(z) = 1 + 0.75z$ has root $z = -\frac{4}{3}$ which is outside the unit circle.

(b) The process is invertible, as is any other AR($p$) process for some finite $p$.

(c) (i) $E(X_t) = \frac{\bar{u}}{1-\varphi_1} = \frac{10}{1+0.75} = \frac{40}{7}$,

(ii) Using formulae (7.47) and (7.46) from Dufour (2008a), we have

$$\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1 - \varphi_1^2} = \frac{16}{7};$$

$$\gamma(k) = \varphi_1^k \gamma(0) = \frac{(-3)^k 16}{4^k 7}.$$ (iii) The autocorrelations are

$$\rho(0) = 1, \quad \rho(k) = \varphi_1^k = \frac{(-3)^k}{4^k}.$$ (d) We can plot $\rho(k)$ for $k = 0, \ldots, 8$:

(e) Write MA($\infty$) representation as $X_t = \psi(B)u_t$ where $\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j$. Because

$$\frac{1}{1 - \varphi_1 B} = 1 + \varphi_1 B + \varphi_1^2 B^2 + \varphi_1^3 B^3 + \varphi_1^4 B^4 + \cdots,$$
we have

\[
\begin{align*}
\psi_0 &= 1, \\
\psi_1 &= \varphi_1 = -\frac{3}{4}; \\
\psi_2 &= \varphi_1^2 = \frac{9}{16}; \\
\psi_3 &= \varphi_1^3 = -\frac{27}{64}; \\
\psi_4 &= \varphi_1^4 = \frac{81}{256}.
\end{align*}
\]

(f) With \(\psi(z) = \frac{1}{1 - \varphi_1 z}\) as defined above, then

\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1})
\]

\[
= \frac{\sigma^2}{(1 - \varphi_1 z)(1 - \varphi_1 z^{-1})}
\]

\[
= \frac{16z}{12 + 25z + 12z^2}.
\]

(g) By Proposition 11.14 from Dufour (2008a), we have

\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(\exp(i\omega)) \psi(\exp(-i\omega))
\]

\[
= \frac{\sigma^2}{2\pi (1 - \varphi_1 \exp(i\omega))(1 - \varphi_1 \exp(-i\omega))}
\]

\[
= \frac{1}{2\pi[1 + 0.75 \exp(i\omega)][1 + 0.75 \exp(-i\omega)]}
\]

\[
= \frac{1}{\pi(3.125 + 3 \cos(\omega))}.
\]

Plotting it yields:

![Graph of the function](image-url)
(h) Using the formula (11) four times, we get
\[ \phi(1) = -\frac{3}{4}, \quad \phi(2) = \phi(3) = \phi(4) = 0. \]

The \( AR(2) \) process \( X_t = 10 + \frac{7}{10}X_{t-1} - \frac{1}{5}X_{t-2} + u_t \)

Write this as
\[ (1 - \varphi_1 B - \varphi_2 B^2)X_t = \bar{\mu} + u_t, \quad \varphi_1 = \frac{7}{10}, \quad \varphi_2 = -\frac{1}{5}. \]

As before, \( u_t \sim N(0, \sigma^2) \) with \( \sigma = 1. \)

(a) Stationarity holds because \( 1 - \varphi_1z - \varphi_2z^2 \) have 2 complex roots that both are outside the unit circle.

(b) Invertibility is immediate because this is an \( AR(2) \) process.

(c) Using the formulas (7.49-51) from Dufour (2008a), we have:

(i) \( \frac{\bar{\mu}}{1-\varphi_1-\varphi_2} = 20; \)
(ii) \[ \rho(0) = 1; \]
(iii) \[ \rho(1) = \frac{\varphi_1}{1-\varphi_2} = \frac{7}{12}, \]
(iv) \[ \rho(2) = \frac{\varphi_1^2 + \varphi_2(1-\varphi_2)}{1-\varphi_2} = \frac{5}{24}, \]
(v) \[ \rho(3) = \varphi_1\rho(2) + \varphi_2\rho(1) = \frac{7}{240}, \]
(vi) \[ \rho(4) = \varphi_1\rho(3) + \varphi_2\rho(2) = \frac{-17}{800}, \]
(vii) \[ \rho(5) = \varphi_1\rho(4) + \varphi_2\rho(3) = \frac{-497}{24000}, \]
(viii) \[ \rho(6) = \varphi_1\rho(5) + \varphi_2\rho(4) = \frac{-2459}{240000}, \]
(ix) \[ \rho(7) = \varphi_1\rho(6) + \varphi_2\rho(5) = \frac{-7273}{2400000}, \]
(x) \[ \rho(8) = \varphi_1\rho(7) + \varphi_2\rho(6) = \frac{-577}{8000000}. \]

In general, for \( k \geq 3 \), we have \( \rho(k) = \varphi_1\rho(k-1) + \varphi_2\rho(k-2) \) and for \( k < 0 \), \( \rho(k) = \rho(-k). \)
(ii) Using formula (7.42) from Dufour (2008a), we have

\[
\gamma(0) = \frac{\sigma^2}{1 - \phi_1 \rho(1) - \phi_2 \rho(2)} = \frac{30}{19}.
\]

For general \(k\), we can easily compute \(\gamma(k) = \rho(k) \gamma(0)\) where \(\rho(k)\) is given above.

(d) Plotting \(\rho(k)\) for \(k = 0, \ldots, 8\) yields

![Plot of \(\rho(k)\) for \(k = 0, \ldots, 8\)](image)

(e) We have

\[
\begin{align*}
\psi_0 &= 1; \\
\psi_1 &= \varphi_1 = \frac{7}{10}; \\
\psi_2 &= \varphi_1^2 + \varphi_2 = \frac{29}{100}; \\
\psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 = \frac{63}{1000}; \\
\psi_4 &= \varphi_1 \psi_3 + \varphi_2 \psi_2 = \frac{-139}{10000}.
\end{align*}
\]

(f) The autocovariance function is

\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1})
\]

where \(\psi(z) = \varphi(z)^{-1}\). In our particular case, the algebra simplifies to

\[
\gamma_x(z) = \frac{100z^2}{(10 - 7z + 2z^2)(2 - 7z + 10z^2)}.
\]

(g)

\[
f_x(\omega) = \frac{\sigma^2}{2\pi[1 - \varphi_1 \exp(i\omega) - \varphi_2 \exp(2i\omega)][1 - \varphi_1 \exp(-i\omega) - \varphi_2 \exp(-2i\omega)]}.
\]
The $MA(2)$ process $X_t = 10 + u_t - 0.75u_{t-1} + 0.125u_{t-2}$

Write this as

$$X_t = \mu + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, \quad \mu = 10, \quad \theta_1 = \frac{3}{4}, \quad \theta_2 = -\frac{1}{8}.$$ 

(a) Stationarity is automatic for all finite-order $MA$ processes.

(b) This $MA(2)$ process is invertible because $\theta(z) = 1 - \theta_1 z - \theta_2 z^2$ has 2 roots 2 and 4 that both are outside the unit circle.

(c) We have:

(i) $E(X_t) = \mu = 10$,

(ii) We have

$$\gamma(0) = \text{Var}(X_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2) = \frac{101}{64}$$
$$\gamma(1) = \sigma^2(-\theta_1 + \theta_1 \theta_2) = -\frac{27}{32},$$
$$\gamma(2) = \sigma^2(-\theta_2) = \frac{1}{8},$$
$$\gamma(3) = \gamma(4) = \cdots = \gamma(8) = 0.$$
(iii) It follows that

\[ \begin{align*}
\rho(0) &= 1; \\
\rho(1) &= \frac{\gamma(1)}{\gamma(0)} = -\frac{54}{101}, \\
\rho(2) &= \frac{\gamma(2)}{\gamma(0)} = \frac{8}{101}, \\
\rho(3) &= \rho(4) = \cdots = \rho(8) = 0.
\end{align*} \]

(d) Plotting \( \rho(k) \) for \( k = 0, \ldots, 8 \) yields

(e) We have

\[ \begin{align*}
\psi_0 &= 1, \\
\psi_1 &= -\theta_1 = -\frac{3}{4}, \\
\psi_2 &= -\theta_2 = \frac{1}{8}, \\
\psi_3 &= 0, \\
\psi_4 &= 0.
\end{align*} \]

(f) The autocovariance generating function is

\[ \begin{align*}
\gamma_x(z) &= \sigma^2 \psi(z) \psi(1/z) \\
&= \sigma^2 (1 - \theta_1 z - \theta_2 z^2) (1 - \theta_1 / z - \theta_2 / z^2) \\
&= \frac{(8 - 6z + z^2)(1 - 6z + 8z^2)}{64z^2}.
\end{align*} \]
(g) The spectral density is
\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(e^{i\omega}) \psi(e^{-i\omega})
= \frac{101 - 108 \cos(\omega) + 16 \cos(2\omega)}{128\pi}.
\]

(h) We have
\[
\phi(1) = -\frac{54}{101}, \quad \phi(2) = -\frac{68}{235}, \quad \phi(3) = -\frac{792}{5177}, \quad \phi(4) = -\frac{208}{2631}.
\]

**The ARMA(1,1) process** \(X_t = 0.5X_{t-1} + u_t - 0.25u_{t-1}\)

Write this as
\[
(1 - \varphi_1 B)X_t = \bar{u} + (1 - \theta_1 B)u_t
\]
where \(\bar{u} = 0, \varphi_1 = 0.5, \theta_1 = 0.25\) and \(u_t \sim N(0, \sigma^2)\) with \(\sigma^2 = 1\).

(a) Stationary: yes because \(1 - \varphi_1 z\) has a single root outside the unit circle.

(b) Invertible: yes because \(1 - \theta_1 z\) has a single root outside the unit circle.

(c) We have:

(i) \(E(X_t) = \frac{\bar{u}}{1-\varphi_1} = 0,\)

(ii) We use formulas (8.39)–(8.41) from Dufour (2008a):
\[
\gamma(0) = (1 - 2\varphi_1 \theta_1 + \theta_1^2) \frac{\sigma^2}{1 - \varphi_1^2} = 13 \frac{12}{12},
\]
\[
\gamma(1) = (1 - \theta_1 \varphi_1)(\varphi_1 - \theta_1) \frac{\sigma^2}{1 - \varphi_1^2} = 7 \frac{24}{24},
\]
and \(\gamma(k) = \varphi_1 \gamma(k-1) = \varphi_1^{k-1} \gamma(1)\) for \(k \geq 2.\)
(iii) We have
\[
\rho(0) = 1,
\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{1 - 2\varphi_1\theta_1 + \theta_1^2}{(1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)} = \frac{7}{26}
\]
and \(\rho(k) = \varphi_1\rho(k-1) = \varphi_1^{k-1}\rho(1)\) for \(k \geq 2\).

(d) Plotting \(\rho(0), \ldots, \rho(8)\) yields

\[\text{Graph showing \(\rho(0), \ldots, \rho(8)\)}\]

(e) We have
\[
\begin{align*}
\psi_0 &= 1, \\
\psi_1 &= \varphi_1 - \theta_1 = \frac{1}{4} \\
\psi_2 &= \varphi_1\psi_1 = \frac{1}{8} \\
\psi_3 &= \varphi_1\psi_2 = \frac{1}{16} \\
\psi_4 &= \varphi_1\psi_3 = \frac{1}{32}
\end{align*}
\]

(f) \(\gamma_x(z) = \sigma^2 \frac{\theta(z)\theta(z^{-1})}{\varphi(z)\varphi(z^{-1})} = \frac{4 - 17z + 4z^2}{8 - 20z + 8z^2}\).

(g) \(f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{\theta[\exp(i\omega)]\theta[\exp(-i\omega)]}{\varphi[\exp(i\omega)]\varphi[\exp(-i\omega)]} = \frac{17 - 8\cos(\omega)}{2\pi(20 - 16\cos(\omega))}\).
(h) Straightforward computation yields:

\[
\begin{align*}
\phi(1) &= \frac{7}{26}, \\
\phi(2) &= \frac{14}{209}, \\
\phi(3) &= \frac{56}{3345}, \\
\phi(4) &= \frac{224}{53521}.
\end{align*}
\]

The ARMA(1, 1) process \( X_t = 0.5X_{t-1} + u_t - 0.5u_{t-1} \)

This is the white noise process in disguise. So it is stationary and invertible. \( \gamma(0) = 1 \) and \( \gamma(k) = 0 \) for \( k \neq 0 \). Similarly, \( \rho(0) = 1 \) and \( \rho(k) = 0 \) for \( k \neq 0 \). Plotting \( \rho(0), \ldots, \rho(8) \) is trivial:
We have $\psi_0 = 1$ and $\psi_k = 0$ for $k \geq 1$. The autocovariance generating function is just $\gamma_x(z) = 1$ whereas the spectral density is the constant $f_x(\omega) = \frac{1}{2\pi}$. Plotting the latter is trivial as well:

Finally, $\phi(1) = \cdots = \phi(4) = 0$ because the white noise can be seen as an $AR(0)$ process.
References

