1. Consider a process that follows the following model: 10 points

\[ X_t = \sum_{j=1}^{m} [A_j \cos(\nu_j t) + B_j \sin(\nu_j t)], ~ t \in \mathbb{Z}, \]

where \( \nu_1, \ldots, \nu_m \) are distinct constants on the interval \([0, 2\pi)\) and \( A_j, B_j, j = 1, \ldots, m \), are random variables in \( L_2 \), such that

\[
E(A_j) = E(B_j) = 0, \ E(A_j^2) = E(B_j^2) = \sigma_j^2, \ j = 1, \ldots, n,
\]

\[
E(A_j A_k) = E(B_j B_k) = 0, \text{ for } j \neq k,
\]

\[
E(A_j B_k) = 0, \ \forall \ j, k.
\]

(a) Show that this process is second-order stationary.

(b) For the case where \( m = 1 \), show that this process is deterministic.

2. Consider the following ARMA model: 40 points

\[ X_t = 0.5 X_{t-1} + u_t - 0.25 u_{t-1} \quad (1) \]

where \( \{u_t : t \in \mathbb{Z}\} \) is an i.i.d. \( N(0,1) \) sequence. Answer the following questions.

(a) Is this model stationary? Why?

(b) Is this model invertible? Why?

(c) Compute:

i. \( E(X_t) \);
ii. \( \gamma(k), k = 1, \ldots, 8; \)

iii. \( \rho(k), k = 1, 2, \ldots, 8. \)

(d) Graph \( \rho(k), k = 1, 2, \ldots, 8. \)

(e) Find the coefficients of \( u_t, u_{t-1}, u_{t-2}, u_{t-3} \) and \( u_{t-4} \) in the moving average representation of \( X_t. \)

(f) Compute the first two partial autocorrelations of \( X_t. \)

(g) If \( X_{10} = 1 \) and assuming the parameters of the model are known, can you compute the best linear forecasts of \( X_{10}, X_{11}, X_{12} \) and \( X_{13} \) based on \( X_{10} \) (only)? If so, compute these.

(h) If \( X_{10} = 1, u_{10} = 2, u_9 = 1, u_8 = 0.99, u_7 = 1.2, \) and assuming the parameters of the model are known, can you compute the best linear forecasts of \( X_{11}, X_{12} \) and \( X_{13} \) based on the history of the process up to \( X_{10} \)? If so, compute these.

3. Let \( X_1, X_2, \ldots, X_T \) be a time series where \( X_1, X_2, \ldots, X_T \) have continuous distributions.

(a) Propose a method for testing the hypothesis that \( X_1, X_2, \ldots, X_T \) are independent and identically distributed (i.i.d.) without any assumption on the existence of the moments for \( X_1, X_2, \ldots, X_T. \)

(b) If \( X_1, X_2, \ldots, X_T \) have common median \( m_0, \) describe a procedure for testing whether these observations are independent without assuming identical distributions.

(c) Consider the “median regression” model:

\[
y_t = x'_t \beta + u_t, \quad t = 1, \ldots, T, \tag{2}\]

where \( x_t, t = 1, \ldots, T, \) are \( k \times 1 \) fixed vectors and the disturbances \( u_t, t = 1, \ldots, T, \) are independent with median zero and continuous distributions. Propose procedures for testing hypotheses of the form \( H_0: \beta = \beta_0 \) and build confidence sets for \( \beta. \)

4. Let \( R_{it}, i = 1, \ldots, n, \) be returns on \( n \) securities for period \( t, \) and \( \tilde{R}_M \) the return on a benchmark portfolio \( (t = 1, \ldots, T). \) The (unconditional) CAPM which assumes time-invariant betas can be assessed by testing:

\[
\mathcal{H}_0: a_i = 0, \quad i = 1, \ldots, n, \tag{3}\]

in the context of the MLR model

\[
r_{it} = a_i + \beta_i \tilde{R}_M + \epsilon_{it}, \quad t = 1, \ldots, T, \quad i = 1, \ldots, n, \tag{4}\]
where \( r_{it} = R_{it} - R_{ft}, \tilde{r}_{Mt} = \tilde{R}_{Mt} - R_{ft} \), \( R_{ft} \) is the riskless rate of return and \( \varepsilon_{it} \) is a random disturbance, such that

\[
V_t = (\varepsilon_{1t}, \ldots, \varepsilon_{nt})' = J W_t, \quad t = 1, \ldots, T,
\]

where \( J \) is an unknown, non-singular matrix and the distribution of the vector \( w = \text{vec}(W) \), \( W = [W_1, \ldots, W_T]' \) is either: (i) known (hence, free of nuisance parameters),
or (ii) specified up to an unknown finite dimensional nuisance-parameter (denoted \( \nu \)).

(a) Put the model (4) in matrix notation.

(b) On assuming that the vectors \( W_1, \ldots, W_T \) are i.i.d. \( N[0, I_n] \), describe the likelihood ratio test for \( \mathcal{H}_E \), and discuss how this test could be implemented.

(c) Propose a procedure for testing whether the errors \( W_1, \ldots, W_T \) are i.i.d. \( N[0, I_n] \).

(d) If another distribution is assumed for \( w \) (such as a heavy-tailed distribution), discuss how such a test could be implemented.

15 points

5. Consider a time series of asset returns \( R_t, t = 1, \ldots, T \), which are i.i.d. according to stable distribution, with characteristic function

\[
\ln \int_{-\infty}^{\infty} e^{ist} d P(S < s) = \begin{cases} 
-\sigma |t|^\alpha [1 - i\beta \text{sign}(t)\tan(\frac{\pi\alpha}{2})] + i\mu t, & \text{for } \alpha \neq 1, \\
-\sigma |t|[1 + i\beta \frac{\pi}{2} \text{sign}(t) \ln |t|] + i\mu t, & \text{for } \alpha = 1.
\end{cases}
\]

(a) Discuss the interpretation of the different parameters \( \mu, \sigma, \alpha \) and \( \beta \).

(b) Why are stable random variables called "stable"?

(c) On assuming that \( \beta = 0 \), propose a method for testing

\[
H_0(\alpha_0): \alpha = \alpha_0.
\]

(d) On assuming that \( \beta = 0 \), discuss how a confidence set for \( \alpha \) could be built.